Feedback stabilization for high order feedforward nonlinear time-delay systems

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Abstract

This paper investigates the problem of global strong stabilization by state feedback, for a family of high order feedforward nonlinear time-delay systems. The uncertain nonlinearities are assumed to satisfy a polynomial growth assumption with an input or delayed input dependent rate. With the help of the appropriate Lyapunov-Krasovskii functionals, and a rescaling transformation with a gain to be tuned online by a dynamic equation, we propose a dynamic low gain state feedback control scheme. A simulation example is given to demonstrate the effectiveness of the proposed design procedure.

Key words: Feedforward systems; time-delay systems; high order; dynamic gain controllers.

1 Introduction

In this paper, we consider the problem of global strong stabilization in the sense of Kurzweil, see [13], for a family of nonlinear systems of dimension $n \geq 2$ described by the following equations:

$$\begin{align*}
\dot{x}_i &= x_{i+1}^p + \phi_i(t, x(t), u(t), x(t-d_i(t))), \\
&\quad i = 1, 2, \cdots, n-1, \\
\dot{x}_n &= u(t),
\end{align*}$$

where $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, time varying delays $d_i(t), i = 1, 2, \cdots, n-1$, satisfying $0 \leq d_i(t) \leq h_i$ and $d_i(t) \leq \tau_i < 1$, with $h_i$ and $\tau_i$ being known constants, and $p \geq 1$ is an odd integer.

Over the last decade, there have been constant progresses on the problem of global stabilization of triangular structural nonlinear systems, see [14], [2] and [8]. One class of triangular structural nonlinear systems are feedback systems, which are also called lower-triangular systems. For the feedback systems of the form

$$\begin{align*}
\dot{x}_i &= x_{i+1}^p + \phi_i(t, x_1, \ldots, x_i), \quad i = 1, \ldots, n-1, \\
\dot{x}_n &= u(t) + \phi_n(t, x_1, \ldots, x_n), \quad y = x_1,
\end{align*}$$

there are many interesting results published recently. For (2) with $p_i = 1, i = 1, 2, \cdots, n-1$, [1] and [9] studied the solvability conditions for the global servomechanism problem and global regulation problem, respectively. For (2) with odd integers $p_i \geq 1, i = 1, 2, \cdots, n-1$, based on the backstepping method, [13] presented a continuous state feedback control law that achieve global strong stability in the sense of Kurzweil.

On the other hand, feedforward systems, which are also called upper-triangular systems, are an important class of nonlinear systems, see [10], [16] and [7]. The past few years have witnessed research efforts towards the development of control schemes for the systems of the form

$$\begin{align*}
\dot{x}_i &= x_{i+1}^p + \phi_i(x_{i+1}, \ldots, x_n, u), \quad i = 1, \cdots, n-1, \\
\dot{x}_n &= u, \quad y = x_1.
\end{align*}$$
When \( p_i = 1, i = 1, 2, \ldots, n - 1, \) [2] considered the feedback stabilization schemes for (3) with functions \( \phi_i \) satisfying

\[
|\phi_i| \leq c (|x|_{i+2} + \cdots + |x|_n),
\]

where \( c \) is a constant, and [16] proposed a universal global state feedback stabilizer design for (3), which does not require a priori knowledge of system nonlinearities. [10] established the first global asymptotic stabilization result for feedforward systems with an homogeneous chain of integrators as approximation at the origin. When odd integers \( p_i \geq 1, i = 1, 2, \cdots, n - 1, \) [15] proposed a design scheme of state feedback controller for (3) with functions

\[
|\phi_i| \leq K (|x|_{i+1} + \cdots + |x|_n) + |u|^{p_i} + |\bar{u}|^{p_i},
\]

where \( K \) is a known constant and \( p_i > p_{i+1} \cdots p_j \), and [12] proposed a design scheme of the low gain output feedback controller for (3) with functions

\[
|\phi_i| \leq a (|x|_{i+2} + \cdots + |x|_n) + |u|^{p_i},
\]

where \( a \) is a known constant.

It is well known that the backstepping method and the forwarding method are powerful tools to design the stability controllers for triangular systems without time-delay, see [14], [9] and [16]. However, the recursive design by constructing either a Krasovskii functional or a Razumikhin function is not a trivial extension of the feedbacking method for the nonlinear non-delay systems, see [5]. In spite of this difficulty, some meaningful results have been presented in the literature for various triangular time-delay systems of the form (1) with \( p = 1, \) [3] studied the output feedback stabilization for a class of stochastic feedback systems which only include output or delayed output nonlinearities. By means of time-varying distributed delay feedback, [6] considered the finite-time global stabilization for a class of feedback systems.

For feedforward time-delay systems, even fewer results have been proposed, and most of them dealt with the systems with time-delay in the input, see [11] and [17]. When nonlinear terms being upper bounded on a compact set by a positive definite quadratic function, [11] solved the problem of globally stabilizing feedforward systems with delayed input. [17] proposed a design scheme of a state feedback controller for a class of input-delayed nonlinear systems that are dominated by an upper-triangular system satisfying the linear growth condition.

To the best of our knowledge, up to now, no work has considered the controller design for high order nonlinear time-delay systems of the form (1). With the help of the appropriate Lyapunov-Krasovskii functional, and a rescaling transformation with a gain to be tuned online by a dynamic equation, this paper will investigate this hard and meaningful problem. The uncertain terms \( \phi_i \)'s in (1) are assumed to be continuous with respect to their arguments and satisfy the following condition.

**Assumption 1.** For all \((f, x_1, \ldots, x_n, u, \bar{x}_1, \ldots, \bar{x}_n, \bar{u}) \in R^+ \times R^{2(n+1)}, \) and \( i = 1, 2, \cdots, n - 1, \) the following conditions hold:

\[
|\phi_i(f, x_1, x_2, \cdots, x_n, u, \bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n, \bar{u})| \\
\leq \rho_i(u) \sum_{j=i+2}^{n+1} |x_j|^{p_i} + |u|^{p_i} + |\bar{u}|^{p_i},
\]

\[
+ \tilde{\rho}_i(\bar{u}) \sum_{j=i+2}^{n+1} |x_j|^{p_i} + |\bar{u}|^{p_i},
\]

where \( x_{n+1} = \bar{x}_{n+1} = 0, \rho_i(\cdot) \) and \( \tilde{\rho}_i(\cdot) \) are known continuous functions with respect to their argument.

**Remark 1.** One can see that (1) satisfying (4) is indeed a feedforward nonlinear system, see [14] and [16]. Compared with [12] and [15], our work has two distinctive features. Firstly, time-delay terms could be seen in our Assumption 1. Secondly the growth rate here is input or delayed input dependent (i.e \( \rho_i(u), \) or \( \tilde{\rho}_i(\bar{u}) \)), while the growth rates in Assumption 3.1 of [12] and Assumptions (1.2b) of [15] are constant. Based on the matrix theory and nonlinear analysis, [18] provided both state feedback controller and output feedback controller for (1) satisfying (4) with \( p = 1 \) and \( y = x_1. \)

In this section, we collect several technical lemmas which are frequently used throughout this paper.

**Lemma 1** [13]. Let \( p \) be a positive integer. Then, for any \( x \in R \) and \( y \in R, \) the following inequalities hold:

\[
|x + y|^p \leq 2^{p-1} |x^p + y^p|,
\]

\[
(|x| + |y|)^\frac{p}{2} \leq |x|^\frac{p}{2} + |y|^\frac{p}{2} \leq 2^{\frac{p-1}{2}} (|x| + |y|)^{\frac{p}{2}}.
\]

If \( p \geq 1 \) is an odd integer, then \( |x - y|^p \leq 2^{p-1} |x^p - y^p|. \)

**Lemma 2** [13]. Let \( c, d \) be positive constants. Given any positive odd integer \( \gamma > 0, \) the following inequality holds:

\[
|x|^d |y|^d \leq \frac{c}{\gamma} x^{\gamma d} + \frac{d+1}{\gamma} y^{\gamma d}.
\]

**Lemma 3.** For any positive odd integer \( p, \) there are two positive constants \( m \) and \( n, \) such that, for any \( x, y \in R^+, \) the inequality \( x^p - y^p \geq (mx - ny)^p \) holds.

We omit the proof of Lemma 3 due to the space limitation.

**Lemma 4.** When \( p \geq 1 \) be an odd integer, for the system

\[
\dot{z}_i = x_i^{p} - l_i \delta z_i, \quad i = 1, 2, \cdots, n - 1,
\]

\[
\dot{z}_n = u - l_n \delta z_n,
\]

where \( \delta \) is any nonnegative function, \( l_1, l_2, \cdots, l_n, \) are positive constants, there is a positive definite and proper Lyapunov function \( V(z), \) a continuous state feedback controller \( u, \) and two positive constants \( \gamma \) and \( \alpha, \) such that

\[
V(i) \leq -\gamma \sum_{i=1}^{\bar{\xi}} \zeta_i^2 - \alpha \cdot \delta \sum_{i=1}^{\bar{\xi}} \zeta_i^2,
\]

where \( \zeta_i \)'s are known nonnegative functions with respect to their argument.
where $V(z)$, $u$ and $\xi_i$ are defined as the following form:

$$\begin{align*}
V(z) &= X^i \\
\gamma_i \gamma_{i+1} \cdots \gamma_n W_i,
\end{align*}$$

$$u(t) = -\beta_n \xi_i^{p-1},$$

$$W_i(z) = (s^{p-1} - z_i^{p-1})^{2-p} \, ds, \quad i = 1, 2, \ldots, n,$$

$$z_i^{p-1} = 0, z_i^{p-1} = -\beta_i - 1 \xi_i - 1, \quad \xi_i = z_i^{p-1} - z_i^{p-1},$$

with $\gamma_i$ and $\beta_i$ being appropriate positive constants. Furthermore, $V$, $u$, $\gamma$ and $\alpha$ are all independent of $\delta$.

**Proof.** From the definitions of $W_i$, we know that,

$$\frac{\partial W_i}{\partial z_i} = |\xi_i|^{2-p-1}, \quad |\partial W_i| \leq a_{i,i} |\xi_i|^{-2}$$

$$V_i(z) = z_i (z_i^p - l_1 \delta z_1) = -l_1 \delta z_i^2 + z_i (z_i^p - z_i^{p-1}) + z_i^2,$$

where constants $a_{i,i} > 0$, $j = 1, 2, \ldots, i - 1$, see [13].

Initial step: Choosing $V_1 = \frac{1}{2} z_1^2$, we can get

$$V_1(z) = z_1 (z_1^p - l_1 \delta z_1) = -l_1 \delta z_1^2 + z_1 (z_1^p - z_1^{p-1}) + z_1^2.$$

Choosing $\gamma_0 = 1, \alpha_1 = l_1, \beta_1 = n$, and letting $z_2^p = -\beta_1 \xi_1$ and $\xi_1 = 1$, we have

$$V_1(z) = -n \gamma_0 \xi_1^2 - \alpha_1 \xi_1^2 + \xi_1 (z_1^p - z_1^{p-1}).$$

Inductive step: Suppose at step $k - 1$, there are positive constants $\alpha_k, \gamma_0, \gamma_{k-1}, \gamma_{k-2}, \beta_1, \beta_2, \ldots, \beta_{k-1}$ which are all independent of $\delta$, such that the $C^1$ positive definite and proper Lyapunov function $V_{k-1}$ satisfying

$$\dot{V}_{k-1}(z) \leq -(n - (k - 2)) \gamma_0 \gamma_{k-1} \cdots \gamma_{k-2} \gamma_{k-1}^{k-1} \xi_1^2$$

$$-\alpha_k \xi_k^2 + \frac{3}{p-1} \xi_k^{p-1} \xi_k^2 + \frac{3}{p-1} \xi_k^{p-1} \xi_k^2,$$

where $z_1^{\ast}, \ldots, z_k^{\ast}, \xi_1, \ldots, \xi_{k-1}$ are defined by (6).

Using (7) and Lemma 2-3, from the tedious calculation, we can find $\alpha_k, \gamma_k-1$ and $\beta_k$, which are all independent of $\delta$, such that $V_k$ satisfying (9) at step $k$, where $z_k^{\ast}$ and $\xi_k$ are defined by (6), and $V_k = \gamma_k V_{k-1} + W_k$, with $W_k$ being $C^1$ (see [13]).

At step $n$ : Letting

$$V_n(z) = \sum_{i=1}^{n} \gamma_i \gamma_{i+1} \cdots \gamma_{n-1} W_i$$

and $\gamma_n = 1$, and $u = -\beta_n \xi_n^{p-1}$, then, constants $\alpha_n$ and $\beta_n$, which are independent of $\delta$, can be found, such that

$$\dot{V}_n(z) \leq -\gamma_0 \gamma_1 \cdots \gamma_{n-1} \xi_1^2 - \alpha_n \delta \xi_1^2.$$

Letting $V = V_n, \gamma = \gamma_0 \gamma_1 \gamma_2 \cdots \gamma_{n-1}$ and $\alpha = \alpha_n$, we can get the result of the lemma.

**Corollary 1.** For (5) with $p = 1$, there is a positive definite matrix $P$, and a state feedback controller $u = -(b_1 z_1 + b_2 z_2 + \ldots + b_n z_n)$, with $b_i$ being positive constants, and two positive constants $\gamma$ and $\alpha$, such that $d^T z^T P z / dt \leq -\gamma ||z||^2 - \alpha \delta ||z||^2$. Furthermore, $P, u, \gamma$ and $\alpha$ are all independent of $\delta$.

**Corollary 2.** For any matrix $D = \text{diag}[l_1, l_2, \ldots, l_n]$, with $l_i$’s being positive constants, there is a positive definite matrix $P$, and positive constants $\alpha, b_i$’s, such that $PB + B^T P \leq -I, PD + DP \geq \alpha I$, with

$$\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-b_1 & -b_2 & -b_3 & \cdots & -b_n
\end{bmatrix}$$

**Remark.** It should be pointed out that Corollary 2 has been widely used in designing dynamic gain controllers for the triangular structural nonlinear systems (2) and (3) with $p_i = 1$, see [1], [7] and [18]. Since Lemma 4 can be viewed as the extension of Corollary 2, Lemma 4 will play a very important role in designing dynamic gain controllers for the triangular structural nonlinear systems (2) and (3) with $p_i \geq 1$.

## 2 State Feedback Controller

**Theorem 1** Under Assumption 1, there exists a continuous state feedback controller of the form

$$u = -\beta_n \sum_{i=1}^{n} \beta_n \xi_i^{p-1} \xi_i^2,$$

where $b_i = \beta_i \beta_{i+1} \cdots \beta_{n-1}, i = 1, 2, \ldots, n - 1$, are constants, $\beta_i$ are defined as in (6), $b_n = 1$, and $L(t)$ is the state of the system

$$L(t) = L_0 \geq 1 \quad \text{for } t \in [-h, 0],$$

with $\varpi(\cdot)$ being an appropriately chosen feedback parameter, $h = \max \{b_1, \ldots, b_{n-1}\}$, and $\alpha$ and $\gamma$ being given in Lemma 4, such that the high order feedforward nonlinear time-delay system (1) is globally strongly stable at $x = 0$. 

3
Proof. It is obvious that the state \( L(t) \) of the system (10) has the following three properties:

\[
\dot{L} \geq 0, \\
\varpi(\cdot) \leq L^{p_i-m} \gamma + \alpha \frac{L}{L^{1-p} - \pi} n^{m-1} z_i,
\]

\( L(t) \geq L(t - d_i(t)) \geq 1, \quad \text{for } t \in [0, +\infty). \) \hfill (11)

These properties will play a very important role in the proof of Theorem 1.

Now we introduce the following rescaling transformation

\[
z_i = L^{-P_n m} x_i, \quad i = 1, 2, \cdots, n,
\]

(12)

where \( L \) is, in contrast to a constant in [12], a dynamic rescaling factor that is updated online by (10). Although rescaling transformation method has been extensively used for (2) and (3) with \( p_i = 1 \), see [1], [7] and [18], up to now, few papers have applied rescaling transformation method for (2) and (3) with \( p_i \geq 1 \).

So the system (1) can be converted into

\[
\begin{align*}
\dot{z}_i &= L^{-P_n m} x_i - (P_n m) \frac{L}{L^{1-p} - \pi} z_i, \\
\dot{z}_n &= L^{-P_n n} u - (P_n) \frac{L}{L^{1-p} - \pi} z_n, \\
\end{align*}
\]

(13)

Using Lemma 4 with \( l_i = P_n m \) and \( \delta = \frac{L}{L^{1-p} - \pi} \), we know that, for the system

\[
\begin{align*}
\dot{z}_i &= z_i^{p_i-1} - (P_n m) \frac{L}{L^{1-p} - \pi} z_i, \\
\dot{z}_n &= u - (P_n) \frac{L}{L^{1-p} - \pi} z_n, \\
\end{align*}
\]

(14)

there is a Lyapunov function \( V(z) \), and a controller \( u(t) \), such that

\[
\dot{V} \leq -\gamma i \xi_i^2 - \alpha i \frac{L}{L^{1-p} - \pi} \xi_i^2,
\]

(15)

where \( \gamma, \alpha, V(z) \) and \( u(t) \) are defined in Lemma 4.

Substituting \( u = -\beta_n \xi_i^{p_i-1} \),

\[
\dot{V} \leq -\beta_n \sum_{i=1}^{n} \xi_i^{p_i-1} - (P_n m) \frac{L}{L^{1-p} - \pi} \xi_i^2,
\]

(16)

into (13), we arrive at

\[
\begin{align*}
\dot{z}_i &= L^{-P_n m} x_i - (P_n m) \frac{L}{L^{1-p} - \pi} z_i, \\
\dot{z}_n &= u - (P_n) \frac{L}{L^{1-p} - \pi} z_n, \\
\end{align*}
\]

(17)

where \( b_i = \beta_i \xi_i^{p_i-1} \), \( i = 1, 2, \cdots, n \), are constants, \( \beta_i \) are defined as in (6), \( b_n = 1 \).

Hence, we can get

\[
\begin{align*}
\dot{V} \leq -L^{-P_n m} \sum_{i=1}^{n} \xi_i^2 - \alpha \frac{L}{L^{1-p} - \pi} \sum_{i=1}^{n} \xi_i^2,
\end{align*}
\]

(18)

Next we will choose an appropriate function \( \varpi(\cdot) \) such that the all states of the system (16) (10) are bounded, and (16), for any function \( L(t) \) generated by (10), is strongly stable at \( z = [z_1, z_2, \cdots, z_n]^T = 0 \).

Proposition 1. Under Assumption 1, there exist continuous and nondecreasing functions \( G_k(\tau) \geq 0, k = 1, 2, \cdots, n - 1 \), such that

\[
G_k(\tau) - \sum_{i=1}^{k} \xi_i^2 \leq -L^{-P_n m} \sum_{i=1}^{n} \xi_i^2 - \alpha \frac{L}{L^{1-p} - \pi} \sum_{i=1}^{n} \xi_i^2,
\]

(19)

With the help of (17) and (18), we can get

\[
\begin{align*}
\dot{V} \leq -L^{-P_n m} \sum_{i=1}^{n} \xi_i^2 - \alpha \frac{L}{L^{1-p} - \pi} \sum_{i=1}^{n} \xi_i^2,
\end{align*}
\]

(20)

Choosing a Lyapunov-Krasovskii functional

\[
V = V + \int_{k-1}^{k} \frac{n-1}{L^{1-p}} \sum_{i=1}^{n} \xi_i^2 \tau_{k-1}^p - \sum_{i=1}^{n} \xi_i^2 ds,
\]

\[
\dot{V} = \frac{n-1}{L^{1-p}} \sum_{i=1}^{n} \xi_i^2 \tau_{k-1}^p - \sum_{i=1}^{n} \xi_i^2 ds,
\]

\[
\dot{V} = \frac{n-1}{L^{1-p}} \sum_{i=1}^{n} \xi_i^2 \tau_{k-1}^p - \sum_{i=1}^{n} \xi_i^2 ds,
\]

(21)

Such that the all states of the system (16) (10) are bounded, and (16), for any function \( L(t) \) generated by (10), is strongly stable at \( z = [z_1, z_2, \cdots, z_n]^T = 0 \).
from (19) and (11), we can get

\[
\dot{V}_{(16)} \leq -L^{-p_n} \gamma i \sum_{i=1}^{n} \xi_i + \alpha i \sum_{i=1}^{n} \xi_i + L^{-p_n} \sum_{k=0}^{n-1} \frac{1}{\tau_k} G_k (|u|) \sum_{i=1}^{n} \xi_i.
\]

Choosing a continuously positive function

\[
\varpi(u) = \sum_{k=0}^{n-1} \frac{1}{\tau_k} G_k (|u|) + \mu,
\]

where \( \mu > 0 \) is a constant, we can get

\[
\dot{V}_{(16)} \leq -L^{-p_n} \gamma i \sum_{i=1}^{n} \xi_i + \alpha i \sum_{i=1}^{n} \xi_i + L^{-p_n} \varpi(u),
\]

where \( \tau_0 = 0 \). Since \( \dot{d}_k(t) \leq \tau_k < 1, k = 1, 2, \ldots, n-1 \), we get

\[
\dot{V}_{(16)} \leq -L^{-p_n} \gamma i \sum_{i=1}^{n} \xi_i + \alpha i \sum_{i=1}^{n} \xi_i + L^{-p_n} \varpi(u).
\]

By (21), \( \dot{V} \) is bounded on \([0, t_f]\), which is the maximal interval of existence of solution of (16) and (10). From the definition of \( \dot{V} \), we can get that the state \( z \) of (16) are bounded on \([0, t_f]\), which implies boundedness of \( u(z) \) in (15). Then we know \( \varpi(u) \) is also bounded on \([0, t_f]\). By the scaling parameter dynamics (10), \( L \) is zero if \( \gamma L^{p_n} \geq \varpi \), so that the boundedness of \( \varpi \) implies boundedness of \( L \). With (21) and \( L \) established by (10), we know that all signals of the system (16) and (10) are bounded on \([0, t_f]\). Hence, \( t_f = \infty \) and solutions of (16) and (10) exist for all time. In view of (21) and Lyapunov-Krasovskii theorem, see [4] and [13], we can conclude that (16) is strongly stable at \( z = 0 \), and hence the closed-loop (13) and (15), for any control gain function \( L(t) \) generated by (10), is strongly stable at \( z = 0 \). Using (15) and (12), we can conclude that (1) can be globally strongly stabilized by a continuous state feedback controller of the form

\[
u = -\beta_n \sum_{i=1}^{n} \frac{1}{\xi_i} L^{-p_n} b_k x_k^{-p_n} i L^{-p_n},
\]

where dynamic gain \( L \) is the state of (10). From (20), we can also know that \( \varpi (\cdot) \) in (10) is dependent on \( \tau_k \), which satisfies \( \dot{d}_k(t) \leq \tau_k < 1 \), and hence the controller (22) is delay-dependent. The proof of Theorem 1 is completed.

3 An Illustrative Example

To show the effectiveness of the control law proposed in Theorem 1, we apply it to the following academic example:

\[
\dot{x}_1 = x_2^3 + \frac{u(t) - \frac{t}{10}}{4}, \quad \dot{x}_2 = u.
\]

(23)

It is easy to verify that Assumption 1 holds for (23).

Following the proof of Theorem 1, one can construct state feedback controller for (23) as follows:

\[
u(t) = -20 L^{-1} x_2^3 + 0.5 L^{-1} x_1^4,
\]

with \( L \) being the state of equation

\[
L = \frac{250}{9} L^2 \max \left\{ 6.5, 1 + u^2 \right\} - 3.3 L^4, 0 \right\}.
\]

(24)

Fig.1 shows the state response of the closed-loop system (23), (24) and (25) with \( d = 5 \), for the initial condition, \( [x_1(t), x_2(t), L(t)] = [-10, 2, 1], t \in [-5, 0] \).

Remark 3. The controllers (22) and (24) are continuous but not smooth. From (24), (25) and Fig.1, one can also find that the gain of controller here is low, which makes the rate of the state of closed-loop system convergence to zero very slow. From more elaborate calculation, one may get more appropriate coefficients of (24) and (25), such that the state \( L \) of (25) becomes lower, and consequently the gain of (24) gets higher to a certain degree. But, in general, the gain of (24) is lower. Indeed, based on the constructive control techniques, see [14], almost all controllers provided for feedforward systems, are always low gain controllers, see [11], [12] and [7].
4 Conclusion

In this paper, a dynamical low gain feedback control is proposed for a family of feedforward nonlinear time-delay systems (1). For (1) with \( p \) being replaced by any odd integers \( p_k \), it is difficult getting reasonable counterparts of (13) and (18) because of the absence of the appropriate state transformation which is the counterpart of (12), and thus we could not study (1) with \( p \) being replaced by any odd integers \( p_k \), which will be a possible future topic. However, we think that Lemma 4 will be a powerful tool to design the stability controllers for triangular structural nonlinear systems.

Appendix. Proof of Proposition 1.

Using Assumption 1, (12) and (11), one can get
\[
P_{\infty} \sum_{i=1}^{n-1} \frac{\partial V}{\partial t} L^{-} \leq \sum_{i=1}^{n-1} |\partial z_{i}| \|\partial z_{i}\|^{p-1} + |u|^{p-1} + \frac{1}{2} \sum_{i=1}^{n-1} |z_{i} - d(t(t))|^{p-1} + |u(t - d(t(t)))|^{p-1},
\]
where \( z_{\infty}(t) = 0 \), for \( t \in [-h, +\infty) \). Noticing (7), we have
\[
P_{\infty} \sum_{i=1}^{n-1} \frac{\partial V}{\partial t} L^{-} \leq \sum_{i=1}^{n-1} |\partial z_{i}| \|\partial z_{i}\|^{p-1} + |u|^{p-1} + \frac{1}{2} \sum_{i=1}^{n-1} |z_{i} - d(t(t))|^{p-1} + |u(t - d(t(t)))|^{p-1},
\]
where \( h_{\infty}(\cdot) \), \( g_{k,i}(\cdot) \), \( i = 1, 2, \ldots, k \), are known continuous functions with respect to their arguments. Then, using (A2), (A3) and (A4), we obtain (18) with \( G_k(\cdot) \), \( k = 0, 1, \ldots, n - 1 \), being known continuous and nondecreasing functions with respect to their arguments.

Acknowledgment

The authors would like to thank all the referees for their constructive and valuable comments.

References