The Kinematic Decoupling of Parallel Manipulators Using Joint-Sensor Data

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Abstract—In this paper we decouple the translational and rotational degrees of freedom of the end-effector of parallel manipulators, and hence, decompose the direct kinematics problem into two simpler sub-problems. Most of the redundant joint-sensor layouts produce a linear decoupling equation expressing the least-square solution of position for a given orientation of the end-effector. The resulting orientation problem can be cast as a linear algebraic system constrained by the proper orthogonality of the rotation matrix. Although this problem is nonlinear, we propose a procedure that provides what we term a decoupled polar least-square estimate. The resulting procedure is fast, robust to measurement noise, and produces estimates with about the same accuracy as a procedure for nonlinear systems if sufficient redundancy is used.

Keywords—Kinematics, decoupling, parallel manipulator, sensor redundancy, least squares.

I. INTRODUCTION

Parallel manipulators consist of two main bodies coupled via $n$ legs. One body is arbitrarily designated as fixed, while the other is regarded as movable, and hence, are respectively called the base and the end-effector (EE) of the manipulator. These manipulators are referred to as Stewart-Gough platforms, when the leg architecture is restricted to an actuated prismatic joint connected to the EE through a passive spherical joint and to the base via a passive universal joint, as shown in Fig. 1. The direct kinematics (DK) problem pertains to the determination of the actual pose—i.e., the position and orientation—of the EE relative to the base from a set of joint-position readings.

Kinematic decoupling refers to the separation of the kinematic relations into two parts: the first part involves only the translational or the rotational motion of the EE; the second part involves both motions. The first part simplifies tremendously the problem because it leads to a linear algebraic system. In this paper we study the kinematic decoupling of the DK problem of a broad class of parallel manipulators using joint-sensor data. We propose a general decoupling equation that allows us to minimize the least-square error in the closure of all kinematic loops, while using joint-sensor redundancy. The various sensor layouts include the particular case where all three Cartesian coordinates of points $B_i$ are measured. Moreover, the concept of isotropic decoupling is discussed, together with the singularity analysis of the proposed procedure.

II. BACKGROUND

In order to describe the motion of the EE relative to the base, let us attach frame $A$ to the base and frame $B$ to the EE, as shown in Fig. 1. The pose of the EE relative to the base is thus determined by $p$, the position vector of the origin of $B$ in $A$, and $R$, the rotation matrix representing the orientation of $B$ in $A$. Alternatively, the said orientation can be represented by $r$, the nine-dimensional array constructed from the three rows of the $3 \times 3$ proper orthogonal matrix $R$ such that

$$R b_i = B_i r,$$

(1)

with

$$B_i = \begin{bmatrix} b_i^T & 0^T & 0^T \\ 0^T & b_i^T & 0^T \\ 0^T & 0^T & b_i^T \end{bmatrix}, \quad R = \begin{bmatrix} r_x^T \\ r_y^T \\ r_z^T \end{bmatrix}, \quad r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}. $$

(2)

Each leg $i$ of a parallel manipulator defines a kinematic loop passing through the origins of frames $A$ and $B$, and through the ankle- and hip-attachment points $A_i$ and $B_i$, respectively.

A. Closure Equation

The error vectors $x_i$ on the closure of each kinematic loop can be expressed in the form

$$x_i = p + R b_i - a_i - q_i, \quad i = 1, \ldots, n,$$

(3)

where $p$ and $R$ are the unknown pose variables and $a_i$ and $b_i$ are the position vectors describing the known geometry of the base and the EE. Moreover, the leg vector
\( q_i \) is only known from the kinematic equation of the leg, together with the readouts of the position sensors located on the different joints along that leg. Since the joints of parallel manipulators are not necessarily all instrumented, we have to eliminate the contribution of the unmeasured joint quantities in the leg kinematics. For this purpose, we use an elimination method introduced earlier by the authors \[1\], which is based on the orthogonal decomposition of the hip-point motion into two parts, one measured and one unmeasured, lying in two orthogonal subspaces of \( \mathbb{R}^3 \). For quick reference, we briefly recall this method below.

The measurement subspace of leg \( i \), denoted \( \mathcal{M}_i \), is defined as the subspace of \( \mathbb{R}^3 \) containing the maximum of the available information from the joint-sensor readouts on the position of the hip-attachment point of the corresponding leg \( i \). Conversely, the subspace orthogonal to \( \mathcal{M}_i \), denoted \( \mathcal{M}_i^\perp \), is called the orthogonal measurement subspace, which is the subspace of \( \mathbb{R}^3 \) containing all the uncertainty associated with this sensor layout. Any vector \( \cdot \) of \( \mathbb{R}^3 \) can be decomposed into two orthogonal parts, \( [\cdot]_M \) and \( [\cdot]_{M^\perp} \), the component lying in \( \mathcal{M}_i \), and \( [\cdot]_{M^\perp} \), the component lying in \( \mathcal{M}_i^\perp \), using the measurement projector \( \mathbf{M}_i \) and an orthogonal complement of \( \mathcal{M}_i \), denoted by \( \mathbf{M}_i^\perp \), as follows:

\[
[\cdot] = [\cdot]_M + [\cdot]_{M^\perp} = \mathbf{M}_i(\cdot) + \mathbf{M}_i^\perp (\cdot). \tag{4}
\]

These two projectors are defined for the four possible dimensions of the measurement subspaces of \( \mathbb{R}^3 \) as:

\[
\mathbf{M}_i = \begin{bmatrix} 1 & P \\ L & O \end{bmatrix}, \quad \mathbf{M}_i^\perp = \begin{bmatrix} O & 3-D measurement \\ L & 2-D measurement \\ P & 1-D measurement \end{bmatrix}, \tag{5}
\]

where \( I \) and \( O \) are the \( 3 \times 3 \) identity and zero matrices, while the plane and line projectors, \( \mathbf{P} \) and \( \mathbf{L} \), are defined as

\[
\mathbf{P} = \mathbf{I} - \mathbf{L}, \quad \mathbf{L} \equiv \mathbf{e}e^T,
\]

in which \( \mathbf{e} \) is a unit vector along line \( \mathcal{L} \) normal to the plane \( \mathcal{P} \). Properly speaking, the identity matrix is not a projector, for the latter is necessarily singular, but we include this matrix in the above list for completeness.

The projection of \( \mathbf{x}_i \) onto \( \mathcal{M}_i \), denoted \( [\mathbf{x}_i]_M \), allows the elimination of the unmeasured part of \( \mathbf{q}_i \), i.e.,

\[
[\mathbf{x}_i]_M = \mathbf{M}_i(\mathbf{p} + \mathbf{Rb}_i - \mathbf{a}_i) - \mathbf{M}_i^\perp \mathbf{q}_i^0. \tag{7}
\]

Vector \( \mathbf{q}_i^0 \) is defined as \( \mathbf{q}_i \), with a value of zero assigned to the position of the unmeasured joint motions. This vector, although different from \( \mathbf{q}_i \), shares the same projection onto \( \mathcal{M}_i \) as \( \mathbf{q}_i \). Moreover, this vector is readily computed with the readouts of position sensors.

\section*{B. Problem Formulation}

The DK problem pertains to the determination of the actual pose of the EE, from a set of joint-position readouts. This problem can be stated as

\[
z = \frac{1}{2} \sum_{i=1}^{n} [\mathbf{x}_i]_M^T [\mathbf{x}_i]_M \rightarrow \min_{\mathbf{p}, \mathbf{R}} \tag{8a}
\]

which is an optimization problem subject to the proper-orthogonality constraint of \( \mathbf{R} \), i.e.,

\[
\mathbf{R}^T \mathbf{R} = \mathbf{I}_{3 \times 3}, \quad \det(\mathbf{R}) = +1, \tag{8b}
\]

with \( [\mathbf{x}_i]_M \) defined in \( (7) \). Apparently, this optimization problem is linear in \( \mathbf{p} \) and \( \mathbf{R} \), but subject to a quadratic constraint on \( \mathbf{R} \), and hence, must be treated as a constrained nonlinear least-square problem. Furthermore, the two unknowns, \( \mathbf{p} \) and \( \mathbf{R} \), are coupled, thereby making it necessary to solve for both simultaneously.

\section*{C. Survey of the State-of-the-Art}

Using only the position of the six prismatic joints as input data, the DK problem of Stewart-Gough platforms can admit up to 40 real possible solutions [2]. The underlying 40\textsuperscript{th}-degree polynomial can be obtained with a procedure proposed by Hust [3]. However, the real-time computation of the 40 roots of this polynomial is prohibitively expensive with current technology, and thus, cannot be used for online implementation. Alternatively, it is well-known that the inverse kinematics of serial manipulators is greatly simplified when the translational and rotational degrees of freedom are decoupled [4]. This kinematic decoupling is obtained when three consecutive revolute axes intersect at a point, a geometric condition fulfilled by most of today’s serial manipulators with their last three joints. Similarly, the DK of parallel manipulators can also be decoupled under some geometric conditions, e.g., the 6-4 architecture [5], the collinearity of five points of \( B \) [6] or the \((3\!-\!1\!-\!1)^2 \) architecture [7]. Although closed-form solution procedures exist for these manipulators, their geometries do not correspond to those of practical use, and hence, an online procedure still needed for those of general geometry. Moreover, all real solutions of the DK problem are equally possible, and thus, it is impossible for the controller to distinguish the actual solution among the set of possible ones with only the six leg lengths. Different approaches have also been proposed to cope with multiple solutions, e.g., the addition of a passive instrumented leg [8], the use of extra sensors on the passive joints [9], or the use of redundant sensors on any of the passive or actuated joints [10]. Here, we use the last approach of redundant sensors because it allows us to determine a decoupling equation from a set of linear kinematics relationships, thereby producing a decoupling of the DK problem without having to impose restrictive geometrical conditions.

\section*{III. Kinematic Decoupling}

The kinematic decoupling thus refers to the separation of the kinematic relations into two parts: the first part involves only the rotational motion of the EE; the second part involves both the rotational and the translational motions. In fact, the second part is a relation between the two unknowns \( \mathbf{p} \) and \( \mathbf{R} \) such that

\[
\mathbf{p} = \mathbf{p}(\mathbf{R}) \tag{9}
\]

The foregoing relation is henceforth called the decoupling equation, while the first part is obtained upon substitution
of (9) into (7). In this section, we propose a decoupling equation based on the closure of all the kinematic loops in order to obtain a complete decoupling of the problem, rather than only partial decoupling, as proposed by Yuan [11].

A. Decoupling Equation

For a given \( \mathbf{R} \), the normality condition of the optimization problem of (8) yields a linear relationship between \( \mathbf{p} \) and \( \mathbf{R} \), which is chosen as the decoupling equation.

**Theorem 1:** Let \( \bar{\mathbf{a}}_M \), \( \bar{\mathbf{B}}_M \) and \( \bar{\mathbf{q}}_M \) be the means of the projection of \( \mathbf{a}_i \), \( \mathbf{B}_i \) and \( \mathbf{q}_i \) on the measurement subspaces \( \mathcal{M}_i \), while \( \bar{\mathbf{M}} \) is the mean of the measurement projectors \( \mathcal{M}_i \). Under a general combination of 3-D, 2-D, 1-D and 0-D measurements of the position of the hip-attachment points, the decoupling equation is given as

\[
\bar{\mathbf{a}}_M + \bar{\mathbf{q}}_M = \bar{\mathbf{M}} \mathbf{p} + \bar{\mathbf{B}}_M \mathbf{r},
\]

where

\[
\bar{\mathbf{a}}_M = \frac{1}{n} \sum_{i=1}^{n} \mathbf{M}_i \mathbf{a}_i, \quad \bar{\mathbf{q}}_M = \frac{1}{n} \sum_{i=1}^{n} \mathbf{M}_i \mathbf{q}_i,
\]

\[
\bar{\mathbf{M}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{M}_i, \quad \bar{\mathbf{B}}_M = \frac{1}{n} \sum_{i=1}^{n} \mathbf{M}_i \mathbf{B}_i.
\]

**Proof:** See the Appendix.

When all the leg-attachment points of the manipulator are under 3-D measurement, we have:

\[
\mathbf{M}_i = \mathbf{I}_{3 \times 3}, \quad \forall \ i = 1, \ldots, n,
\]

the closer equation (7) becomes identical to (3) and Theorem 1 reduces to

**Corollary 1:** Let \( \mathbf{a}_o \), \( \mathbf{b}_o \) and \( \mathbf{q}_o \) be the mean values of the sets \( \{ \mathbf{a}_i \}^n \), \( \{ \mathbf{b}_i \}^n \) and \( \{ \mathbf{q}_i \}^n \), respectively. Under a 3-D measurement of the position of the leg hip-attachment points, the decoupling equation is given as

\[
\mathbf{a}_o + \mathbf{q}_o = \mathbf{p} + \mathbf{R} \mathbf{b}_o,
\]

where

\[
\mathbf{a}_o = \frac{1}{n} \sum_{i=1}^{n} \mathbf{a}_i, \quad \mathbf{q}_o = \frac{1}{n} \sum_{i=1}^{n} \mathbf{q}_i, \quad \mathbf{b}_o = \frac{1}{n} \sum_{i=1}^{n} \mathbf{b}_i.
\]

**Proof:** See the Appendix.

For 3-D measurements, we refer to Fig. 2, showing the centroids of the leg-attachment points as joined by vector \( \mathbf{q}_o \). If now the origins of frames \( A \) and \( B \) are chosen at the centroids of the foregoing points, \( \{ A_i \}^n \) and \( \{ B_i \}^n \), then \( \mathbf{a}_o = \mathbf{b}_o = \mathbf{0} \), and the 3-D decoupling equation reduces to

\[
\mathbf{p} = \mathbf{q}_o,
\]

where \( \mathbf{R} \) does not appear any more. In this case, \( \mathbf{p} \) can be solved for independent of \( \mathbf{R} \), meaning that we have obtained a complete decoupling of the DK problem.

Now, we reduce the DK problem to an orientation problem by eliminating the position with the decoupling equation. For 3-D measurements, the position elimination is equivalent to locating the centroid of \( \{ B_i \}^n \) at the centroid of \( \{ A_i \}^n \) plus the mean “leg-vector” \( \mathbf{q}_o \), while the orientation problem corresponds to rotating the EE around the centroid of \( \{ B_i \}^n \).

B. Position Elimination

The unknown position \( \mathbf{p} \) is eliminated from the kinematic relations by substituting (10) into (7), i.e.,

\[
[x_i]_M = M_i (\bar{\mathbf{B}}_i \mathbf{r} - \bar{\mathbf{a}}_i - \bar{\mathbf{q}}_i),
\]

where \( \bar{\mathbf{B}}_i \), \( \bar{\mathbf{a}}_i \) and \( \bar{\mathbf{q}}_i \) are defined as

\[
\bar{\mathbf{B}}_i \equiv \mathbf{B}_i - \bar{\mathbf{M}}^{-1} \bar{\mathbf{B}}_M, \quad \bar{\mathbf{a}}_i \equiv \bar{\mathbf{a}}_i - \bar{\mathbf{M}}^{-1} \bar{\mathbf{a}}_M, \quad \bar{\mathbf{q}}_i \equiv \bar{\mathbf{q}}_i - \bar{\mathbf{M}}^{-1} \bar{\mathbf{q}}_M.
\]

Apparently, the change of variables of (14b) requires the inversion of \( \bar{\mathbf{M}} \), henceforth called the decoupling matrix. Since the accuracy of the decoupling depends on the condition number of this matrix, we discuss in section IV the conditions allowing us to obtain an isotropic decoupling, i.e., an optimally-conditioned matrix \( \bar{\mathbf{M}} \) of the DK problem. When all the leg-attachment points of the manipulator are under 3-D measurement, (14) reduces to (15), i.e.,

\[
[x_i]_{3-D} = \mathbf{R} \mathbf{b}'_i - \mathbf{a}'_i - \mathbf{q}'_i,
\]

where \( \mathbf{b}'_i \), \( \mathbf{a}'_i \) and \( \mathbf{q}'_i \) are defined as

\[
\mathbf{b}'_i \equiv \mathbf{b}_i - \mathbf{b}_o, \quad \mathbf{a}'_i \equiv \mathbf{a}_i - \mathbf{a}_o, \quad \mathbf{q}'_i \equiv \mathbf{q}_i - \mathbf{q}_o.
\]

Note that, in this case, the decoupling matrix is simply the \( 3 \times 3 \) identity matrix.

C. Orientation Problem

**General Measurements**

The unconstrained least-square (ULS) estimate of \( \mathbf{r} \), denoted \( \hat{\mathbf{r}} \), can be obtained by relaxing the orthogonality constraint of (8b), while solving the orientation problem of (8a)
with \([x_i]_M\) defined as in (14). This estimate is found as the LS solution of a \(3n \times 9\) linear algebraic system, with \(n \geq 3\), namely,

\[
B_M' \hat{r} = (a'_M + q'_M),
\]

(16a)

where matrix \(B_M'\) and vectors \(a'_M\) and \(q'_M\) are defined as

\[
B_M' \equiv \begin{bmatrix}
M_1 \bar{b}_1 \\
\vdots \\
M_n \bar{b}_n
\end{bmatrix},
\]

(16b)

\[
a'_M \equiv \begin{bmatrix}
M_1 \bar{a}_1 \\
\vdots \\
M_n \bar{a}_n
\end{bmatrix}, \quad q'_M \equiv \begin{bmatrix}
M_1 \bar{q}_1 \\
\vdots \\
M_n \bar{q}_n
\end{bmatrix}.
\]

The ULS estimate \(\hat{r}\) of (16) is symbolically obtained via the generalized inverse of \(\hat{B}_M', \hat{B}_M'^\dagger\), namely,

\[
\hat{r} = \hat{B}_M'^\dagger (a'_M + q'_M), \quad \hat{B}_M'^\dagger = (\hat{B}_M'T \hat{B}_M')^{-1} \hat{B}_M'T.
\]

(17)

Due to unavoidable noise and the assumption of independence of the nine components of \(\hat{R}\), the ULS estimate \(\hat{R}\) may not be orthogonal. The closest orthogonal estimate is obtained as the polar projection\(^1\) of \(\hat{R}\) over the constraint manifold of (8b). This second estimate, henceforth called the decoupled polar least-square (DPLS) estimate, and denoted \(\hat{R}\), is computed as the orthogonal factor of the polar decomposition of \(\hat{R}\), i.e.,

\[
\hat{R} = \hat{Q}, \quad \hat{R} = \hat{Q} \hat{W},
\]

(18)

where the \(3 \times 3\) matrices \(\hat{Q}\) and \(\hat{W}\) are, respectively, orthogonal and either positive-semidefinite or positive-definite. Once \(\hat{R}\) is known, \(\hat{p}\) can be computed as the LS approximation of

\[
M \hat{p} = (a_M + q_M - \bar{b}_M),
\]

(19a)

with

\[
M \equiv \begin{bmatrix}
M_1 \\
\vdots \\
M_n
\end{bmatrix}, \quad \bar{b}_M \equiv \begin{bmatrix}
M_1 Rb_1 \\
\vdots \\
M_n Rb_n
\end{bmatrix},
\]

(19b)

\[
a_M \equiv \begin{bmatrix}
M_1 \bar{a}_1 \\
\vdots \\
M_n \bar{a}_n
\end{bmatrix}, \quad q_M \equiv \begin{bmatrix}
M_1 \bar{q}_1 \\
\vdots \\
M_n \bar{q}_n
\end{bmatrix}.
\]

\(^1\) In the plane, a polar projection can be geometrically interpreted as the projection of a point of the plane onto the unit circle centered at the origin of the plane; this projection is the intersection of the circle with the line joining the point with the center of the circle. Obviously, of the two intersections, the one closest to the point is chosen.

3-D measurements

When all leg-attachment points are under 3-D measurement, the decoupled DK problem of (8) can be recast as an optimization problem, namely,

\[
z = \frac{1}{2} \text{tr}(X^T X) \rightarrow \min_{\hat{X}} \quad (20a)
\]

\[
\hat{X} = RB - A
\]

(20b)

subject to (8b), with the \(3 \times n\) matrices \(X, B\) and \(A\) defined as

\[
X \equiv [x_1]_{3-D} \ldots [x_n]_{3-D} ,
\]

(20c)

\[
B \equiv [b'_1 \ldots b'_n], \quad A \equiv [a'_1 + q'_1 \ldots a'_n + q'_n].
\]

This optimization problem is well known in the field of computational statistics as the orthogonal Procrustes problem [13]. Although the foregoing objective function is quadratic in \(\hat{R}\), it is possible to transform it into a linear form with the aid of an equality constraint. Indeed, upon substitution of (20b) into (20a), and simplifying the second term of the right-hand side (RHS), gives:

\[
z = \frac{1}{2} \text{tr}(A^T A) + \frac{1}{2} \text{tr}(B^T B) - \text{tr}(A^T RB). \quad (21)
\]

Now, using the properties of the trace operator, the third term of the RHS of (21) can be rearranged as

\[
\text{tr}(A^T RB) = \text{tr}(BA^T R). \quad (22)
\]

Therefore, the quadratic objective function of (21) reduces to the linear form:

\[
z = \frac{1}{2} \text{tr}(A^T A) + \frac{1}{2} \text{tr}(B^T B) - \text{tr}(BA^T R), \quad (23)
\]

still constrained by the proper orthogonality of \(\hat{R}\).

Since the unknown, \(\hat{R}\), appears only in the last term, which is negative, the minimization of \(z\), as given in (23), is equivalent to the maximization of its last term alone, i.e.,

\[
\zeta = \text{tr}(BA^T R) \rightarrow \max_{\hat{R}} \quad (24)
\]

subject to (8b). This quadratically-constrained least-square problem can be readily solved by resorting to the result below:

**Lemma 1:** Let \(C\) and \(R\) be two \(n \times n\) matrices with \(C = QW\), where \(Q\) is orthogonal and \(W\) is symmetric and positive-semidefinite. Moreover, let

\[
\zeta = \text{tr}(C^T R) \rightarrow \max_{\hat{R}} \quad (25a)
\]

subject to

\[
R^T R = 1 \quad (25b)
\]

The maximum of \(\zeta\) is given by \(R = Q\).

**Proof:** See the Appendix.

By virtue of Lemma 1, the solution of (24), namely, the DPLS estimate \(\hat{R}\), is readily found to be

\[
\hat{R} = \hat{Q}, \quad AB^T = \hat{Q} W, \quad (26)
\]

where the orthogonal matrix \(\hat{Q}\) and the symmetric positive-semidefinite matrix \(W\) are obtained as the two factors of the polar decomposition of the product \(AB^T\). Once \(\hat{R}\) is known, (19) can be used to solve for \(\hat{p}\).
IV. Singularity Analysis

It is recalled that joint-sensor redundancy is used here to cope with both the measurement noise and the nonlinearity of the problem at hand, and hence, the procedure is meant for use with 9 to 18 sensors. There are several conditions under which the procedure does not provide an estimate of the actual pose of the end-effector, as we discuss below.

First, the kinematic decoupling of (10a) does not exist when the measurement subspaces \( \{ M_i \}^n_i \) do not span \( \mathbb{R}^3 \). In fact, the change of variable of (14) requires the inversion of the decoupling matrix \( \tilde{M} \). The accuracy of the decoupling depends on the condition number of \( \tilde{M} \), i.e.,

\[
\kappa(\tilde{M}) = \frac{\sigma_3}{\sigma_1}, \quad \text{where} \quad \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0, \tag{27}
\]

where \( \{ \sigma_i \}^3_i \) are the singular values of \( \tilde{M} \). This number expresses “how well” the set of measurement subspaces \( \{ M_i \}^n_i \) span \( \mathbb{R}^3 \). Matrices with the optimum condition number of unity are called isotropic, an isotropic decoupling being thus one that provides equal accuracy in all directions. In an earlier work [12], we proposed necessary, but not sufficient, geometrical conditions for isotropic decoupling. For example, we showed that isotropy is reachable with the geometry of an industrial Stewart-Gough manipulator if the yoke joints are oriented on the base in a particular way. In the case of 3-D measurements, the decoupling matrix is the identity, and hence, the decoupling always exists and is, additionally, isotropic.

Second, the full rank of \( B_M \) cannot be guaranteed without considering the manipulator geometry, the location of the EE and the joint-sensor layout. The singularity analysis of \( B_M \) can be conducted along the same lines of that for Jacobian matrices. The solution of the orientation problem of (16) requires the full rank of \( B_M \), i.e., \( \text{rank} (B_M) \geq 9 \), and hence, the use of a minimum of nine sensors. However, the nine unknowns of \( R \) are not independent and can be reduced to a minimum of four unknowns if necessary, the others being rather computed a posteriori with (8b). This approach is necessary when the geometry of the end-effector is made of coplanar hip-attachment points, as is the case of Stewart-Gough platforms. Assume that these points are in the \( x-y \) plane of frame \( B \). The \( z \) component of \( \mathbf{b}_z \) is zero, and hence, it is not possible to solve for the 3rd, 6th and 9th components of \( \mathbf{r} \) with (16). In this case, it is mandatory to eliminate the corresponding columns in \( B_M \) and then solve a posteriori for the remaining components of \( \mathbf{r} \). In the case of 3-D measurements, a similar situation arises when the hip-attachment points are coplanar. The third row of \( B \) is zero, and hence, it is not possible to solve for the 3rd column of \( R \), i.e. the 3rd, 6th and 9th components of its equivalent \( \mathbf{r} \). In this case, one or even a few columns can be added to matrices \( A \) and \( B \) in order to render them of full rank. An additional column \( k \) can be derived from the columns \( i \) and \( j \) of \( A \) and \( B \) as \( (a'_i + c'_i) \times (a'_j + c'_j) \) and \( b'_k = b'_i \times b'_j \).

The performance of each of the two versions of the DPLS solution procedure is now assessed.

V. Performance Assessment

In order to evaluate the performance of the DPLS procedure, we consider two performance indices: the computational cost of the procedure and the accuracy of their estimates while using noisy input data. The DPLS procedure is compared with a traditional constrained least-square (CLS) procedure requiring an initial guess together with an iterative technique based on Newton’s method in order to converge toward its CLS estimate. It is noteworthy that the computation of the DPLS estimate \( (\mathbf{p}, \mathbf{R}) \) of the actual solution \( (\mathbf{p}, \mathbf{R}) \) of the DK problem does not require iterations as for the computation of the CLS estimate.

A. Computational Cost

Let us denote by \( (\cdot)_g \) and \( (\cdot)_b \) the two versions of the procedures under general and 3-D position measurements of all the hip-attachment points of a parallel manipulator of general geometry. The computational cost of these procedures is measured with the number of floating-point operations—flops—using the assumptions below:

1. The generalized inverse is computed via Householder reflections and back-substitutions of a \( p \times q \) linear algebraic system with \( p \geq q \), which requires \( q^2 (p - q/3) \) flops [13];
2. The inversion of the symmetric \( 3 \times 3 \) decoupling matrix requires 41 flops in general and no flops when isotropic;
3. The orthogonal factor of the polar decomposition is computed via Higham’s polar-decomposition algorithm [14], which requires 59 flops per iteration and an average of 3 iterations, for a total of 177 flops;
4. The origin of frames \( A \) and \( B \) is chosen at the centroid of \( \{ A_i \}^n_i \) and \( \{ B_i \}^m_i \) in order to avoid the calculation of \( \mathbf{a}_o = \mathbf{b}_o = \mathbf{0} \);
5. The number of instrumented legs \( n \) is set to six.

As shown in Fig. 3, the computational cost of the CLS procedure increases with the number of iterations, while the cost of the DPLS procedure remains constant. For any joint-sensor layout, the actual computational cost of the procedures is between the upper bound \( (\cdot)_g \) and the lower bound \( (\cdot)_b \). Moreover, the computational cost is further

![Fig. 3. Computational cost of the various solution procedures](image-url)
reduced with some specific geometries. With general measurements, the DPLS$_g$ procedure is more efficient than the CLS$_g$ procedure if the latter requires more than three iterations. With 3-D measurements, the DPLS$_3$ procedure is always more efficient than the CLS$_3$ procedure. Apparently, the kinematic decoupling produces a strong reduction of flops with 3-D measurements, which makes it attractive for online computations.

B. Estimation Accuracy

The estimation accuracy is measured with the position and orientation errors between the estimate and the actual pose, $\delta$ and $\epsilon$, respectively, which are defined below:

$$\delta \equiv \|\mathbf{p} - \hat{\mathbf{p}}\|, \quad \epsilon \equiv \|\text{vect}(\mathbf{R}^T \mathbf{R})\|,$$

(28)

where $\text{vect}(\cdot)$ is the axial vector of its $3 \times 3$ matrix argument [17]. The geometry of the manipulator is chosen as the Stewart-Gough platform used in 300-series flight simulators produced by CAE Electronics Ltd., with a scale factor of 1:50. The noisy input data are obtained either directly from measurements on an experimental prototype or from simulations by adding a Gaussian zero-mean noise to the theoretical joint readouts; the noisy data are then submitted to each of the DPLS and CLS procedures in order to compute their estimates. For simulation, the actual poses of the EE are known from the prescribed trajectory; for experiments, these poses have to be computed from very accurate measurements taken with a coordinate measuring machine (CMM). Shown in Fig. 4 is the experimental prototype, manufactured with rather loose tolerances. Moreover, the joints of the prototype operate under no feedback loop. The EE is held at a specific pose by a stand placed between the two end-plates. A single instrumented leg is used, that is manually placed at each of the six leg locations. Potentiometers are installed on the first three joints. The output voltage of these sensors is sent to a multi-channel 12-bit A/D converter. The resulting digital values are the noisy input data to each of the estimation procedures.

<table>
<thead>
<tr>
<th>Sensors</th>
<th>Type</th>
<th>3-D measurements for 3-D measurements</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>A</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
</tr>
<tr>
<td>15</td>
<td>B</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
</tr>
<tr>
<td>12</td>
<td>C</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
</tr>
<tr>
<td>9</td>
<td>D</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
</tr>
<tr>
<td>9</td>
<td>E</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
</tr>
<tr>
<td>9</td>
<td>F</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
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<td>9</td>
<td>G</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
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<tr>
<td>9</td>
<td>H</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
</tr>
<tr>
<td>9</td>
<td>I</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
</tr>
<tr>
<td>9</td>
<td>J</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
</tr>
</tbody>
</table>

$\hat{\delta}$: 3-D measurement of the position $i$ and attachment point of leg $i$.

$\hat{i}$: Unmeasured position of the hip-attachment point of leg $i$.

Fig. 5. Relative position $\hat{\delta}$ (mm) and orientation $\hat{i}$ (mrad) errors of the CLS$_3$ and DPLS$_3$ estimates.
different distributions of sensors arise. Layouts C to F, that use 4 legs × 3 sensors/leg = 12 sensors, produce different levels of accuracy, because of the different distributions of the sensors. Moreover, layout G, which uses only 9 sensors, produces a better accuracy than layout C, which uses 12 sensors. Indeed, the good distribution of sensors in layout G enhances its accuracy.

**General measurements**

With general measurements, the linear DPLSg procedure does not produce identical estimates as the nonlinear CLSg procedure, and hence, both procedures have different accuracies, while using the same set of noisy input data. A nine-sensor layout is used on the experimental prototype, i.e., \( \{ \tilde{R}RP, \tilde{R}RP, \tilde{R}RP, \tilde{R}RP, \tilde{R}RP, \tilde{R}RP \} \), where \( \tilde{R}RP \) denotes the presence of a position sensor on the first two revolute joints and no sensor on the prismatic joint, while \( \tilde{R}RP \) denotes the presence of a position sensor on the first revolute joint only.

The linear DPLS estimate must be considered as a good alternative to the nonlinear CLS estimate, because although different from the latter, it provides estimates of the actual solution with about the same accuracy as the latter without requiring iterative computations.

**Acknowledgments**

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**Appendix**

**Theorem 1:**

*Proof:* Upon substitution of (7) and (11) into (8a), the decoupling equation is derived as the normality condition of \( z \) with respect to \( p \), i.e.,

\[
\frac{dz}{dp} = \left( \sum_{i=1}^{n} M_i p \right) + \left( \sum_{i=1}^{n} M_i B_i \right) r
\]

\[
- \left( \sum_{i=1}^{n} M_i a_i \right) - \left( \sum_{i=1}^{n} M_i q_i^0 \right) = 0,
\]

from which (10) of Theorem 1 is readily obtained. This normality condition yields a minimum when the Hessian matrix of \( z \) with respect to \( p \), i.e.,

\[
\frac{d^2 z}{d p^2} = \sum_{i=1}^{n} M_i = n \hat{M},
\]

is strictly positive-definite. This condition arises when the direct sum of the \( n \) measurement subspaces, i.e., \( M_1 \oplus \ldots \oplus M_n \), yields \( \mathbb{R}^3 \).

**Corollary 1:**

*Proof:* Upon substitution of (7) and (11) into (8a), the 3-D decoupling equation is derived as the normality condition of \( z \) with respect to \( p \), i.e.,

\[
\frac{dz}{dp} = np + R \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} q_i = 0,
\]

from which (12) of Corollary 1 is readily obtained. This condition leads always to a minimum because the Hessian matrix of \( z \) with respect to \( p \), i.e.,

\[
\frac{d^2 z}{d p^2} = n I_{3 \times 3},
\]

is the sum of \( n \geq 1 \) identity matrices, the result thus being always symmetric and strictly positive definite.

**Lemma 1:**

*Proof:* Let us introduce the \( n \times n \) orthogonal matrix \( Z \) such that:

\[
Z = Q^T R
\]

Then (25a) reduces to

\[
\zeta = \text{tr}(C^T R) = \text{tr}(W^T Q^T R) = \text{tr}(W^T Z)
\]
Since \( \mathbf{W} \) is symmetric and positive-semidefinite, it has non-negative eigenvalues \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 \), and the corresponding unit eigenvectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are mutually orthogonal. Hence \( \mathbf{W} \) can be represented as

\[
\mathbf{W} = \mathbf{W}^T = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \mathbf{v}_i^T, \tag{35}
\]

which is also known as the spectral decomposition of \( \mathbf{W} \). By virtue of the invariance of the trace under a cyclic change of the factors of a matrix product, i.e., \( \text{tr}(\mathbf{ABC}) = \text{tr}(\mathbf{BCA}) = \text{tr}(\mathbf{CAB}) \), and upon substitution of (35) into (34), we obtain:

\[
\zeta = \sum_{i=1}^{3} \lambda_i \text{tr}(\mathbf{v}_i \mathbf{v}_i^T \mathbf{Z}) = \sum_{i=1}^{3} \lambda_i \text{tr}(\mathbf{v}_i^T \mathbf{Z} \mathbf{v}_i) = \sum_{i=1}^{3} \lambda_i (\mathbf{v}_i, \mathbf{Z} \mathbf{v}_i), \tag{36}
\]

where \((\mathbf{v}_i, \mathbf{Z} \mathbf{v}_i)\) denotes the inner product of vectors \(\mathbf{v}_i\) and \(\mathbf{Z} \mathbf{v}_i\). The three inner products of (36) are bounded by the Cauchy-Schwarz inequality as

\[
(\mathbf{v}_i, \mathbf{Z} \mathbf{v}_i) \leq \|\mathbf{v}_i\| \|\mathbf{Z} \mathbf{v}_i\| = 1, \tag{37}
\]

where the maximum value of unity is reached when the orthogonal matrix \(\mathbf{Z}\) is equal to the identity matrix. Using (33), the maximum of the objective function of (36) is reached for

\[
\mathbf{Z} = 1 = \mathbf{Q}^T \mathbf{R}, \tag{38}
\]

and hence, \(\mathbf{R} = \mathbf{Q}\).

### References


