7

Overview of Stability Analysis Methods

In this chapter we briefly review some of the main tools for certifying the stability of a dynamical system model, especially in the multidimensional case. We consider both the state-space (Lyapunov) and input-output modeling point of view.

7.1 Lyapunov Stability of State-Space Systems

7.1.1 Continuous-Time Lyapunov Stability

Consider the nonlinear autonomous dynamical system:

$$\dot{x}(t) = f(x(t)), \;\; x(0) = x_0, \;\; t \geq 0, \tag{7.1}$$

where $x(t) \in \mathcal{D} \subset \mathbb{R}^n$ for all $t \geq 0$, and $\mathcal{D}$ is an open subset containing 0. We assume for simplicity in the following that for all x_0 of interest, a unique solution exists for (7.1) over the interval $[0, \infty)$. Moreover, we assume $f(0) = 0$, i.e., $x(t) \equiv 0$ is an equilibrium solution, and consider stability notions for this equilibrium.

Definition 7.1.1. Assume that $f(0) = 0$. Then the zero equilibrium point is

1. *Lyapunov stable* if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $|x(0)| < \delta$, then $|x(t)| < \epsilon$, for all $t \geq 0$.

2. *unstable* if it is not Lyapunov stable.

3. *(locally) asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $|x(0)| < \delta$, then $\lim_{t\to\infty} x(t) = 0$.

4. *(locally) exponentially stable* if there exist positive constants α, β and δ such that if $|x(0)| < \delta$, then $|x(t)| \leq \alpha |x(0)| e^{-\beta t}, t \geq 0$.

5. *globally asymptotically stable* if it is Lyapunov stable and for all $x(0) \in \mathbb{R}^n$, $\lim_{k\to\infty} x(t) = 0$.

6. *globally exponentially stable* if there exist positive constants α, β such that $|x(t)| \leq \alpha |x(0)| e^{-\beta t}, t \geq 0$, for all $x(0) \in \mathbb{R}^n$.

Theorem 7.1.1 (Lyapunov's direct method, continuous-time)**.** *Consider the continuous-time system (7.1), with $f(0) = 0$, and assume that there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\begin{aligned}
&V(0) = 0 \\
&V(x) > 0, \forall x \in \mathcal{D} \setminus \{0\}, \\
&\frac{d}{dt} V(x) = \dot{V}(x) \equiv \left[\frac{\partial V}{\partial x}(x)\right]^T f(x) \leq 0, \forall x \in \mathcal{D}.
\end{aligned}$$

Then the zero equilibrium point is Lyapunov stable. If in addition

$$\dot{V}(x) < 0, \forall x \in \mathcal{D} \setminus \{0\}, \tag{7.2}$$

then the zero equilibrium point is asymptotically stable. It there exist scalars $\alpha, \beta, \epsilon > 0$*, and* $p \geq 1$ *such that* V *satisfies*

$$\alpha |x|^p \leq V(x) \leq \beta |x|^p, \quad \forall x \in \mathcal{D}, \tag{7.3}$$

$$\dot{V}(x) \leq -\epsilon V(x), \quad \forall x \in \mathcal{D}, \tag{7.4}$$

then the zero equilibrium point is exponentially stable, and in fact

$$|x(t)| \leq \left(\frac{\beta}{\alpha}\right)^{1/p} |x(0)| e^{(-\epsilon/p)t}, \quad t \geq 0.$$

Finally if V *is radially unbounded, i.e.,*

$$V(x) \to \infty \;\; as \; |x| \to \infty,$$

then (7.2) with $\mathcal{D} = \mathbb{R}^n$ *implies that the zero equilibrium point is globally asymptotically stable. Similarly, if (7.3), (7.4) hold for* $\mathcal{D} = \mathbb{R}^n$*, then the zero equilibrium point is globally exponentially stable.*

In addition to Lyapunov's direct method for stability analysis, linearization of a nonlinear system around an equilibrium can also provide local stability information about this equilibrium, using the following important theorem. Here $\text{Spec}(A)$ denotes the spectrum of the matrix A, i.e., its set of eigenvalues.

Theorem 7.1.2 (Lyapunov's indirect method)**.** *Consider the nonlinear system (7.1), with* f *continuously differentiable. Let* $A = \frac{\partial f}{\partial x}|_{x=0}$ *be the Jacobian of* f *at the equilibrium. Then the following holds*

1. *If Re* $\lambda < 0$ *for all* $\lambda \in$ *Spec*(A)*, then the zero solution is (locally) exponentially stable.*

2. *If Re* $\lambda > 0$ *for some* $\lambda \in$ *Spec*(A)*, then the zero solution is unstable.*

Application to Linear Systems and Quadratic Stability

Consider a CT linear system

$$\dot{x}(t) = Ax(t), \; x(0) = x_0, t \geq 0. \tag{7.5}$$

If $\det(A) \neq 0$, 0 is the unique equilibrium of this system. If $\det(A) = 0$, then every point in the null space of A is an equilibrium, not just 0 (note that a linear system can never have multiple isolated equilibria). We would like to study the stability of the 0 equilibrium. It is well known that asymptotic stability of this equilibrium is equivalent to A being Hurwitz, i.e., having all its eigenvalues with strictly negative real part. In this case, we have in fact global exponential stability. If some eigenvalues have 0 real part, we can still get Lyapunov stability (but not asymptotic stability) if all Jordan blocks for these eigenvalues in the Jordan decomposition of A have size 1, i.e., are simply of the form $J = \lambda_i$. If any eigenvalue has strictly positive real part, the system is unstable.

Stability of (7.5) also turns out to be *equivalent to quadratic stability*, i.e., the existence of a quadratic Lyapunov function $V(x) = x^T P x$, for some $P \succ 0$.[1] Writing the condition (7.2), we obtain

$$A^T P + PA \prec 0, \tag{7.6}$$

called a continuous-time Lyapunov inequality. To obtain exponential stability with an explicit bound ϵ on the convergence rate, the inequality (7.12) translates into

$$A^T P + PA \prec -\epsilon P, \; \epsilon > 0.$$

[1]The notation $P \succ 0$ means that P is positive definite, $P \succeq 0$ that it is positive semi-definite. $A \succ B$ means $A - B \succ 0$, etc.

In this case, we can find a matrix P satisfying these conditions by solving a CT Lyapunov equation, which is a linear equation in P of the form

$$A^T P + PA = -Q, \text{ where } Q \succ 0. \tag{7.7}$$

See the MATLAB command `lyap`.

Alternatively, (7.6) is a *linear matric inequality* (LMI) and matrices P satisfying it can be found directly using semi-definite programming (for example, the Matlab toolboxes `cvx` or `YALMIP` let you express and solve such inequalities nicely). An LMI in the variable X is an inequality of the form

$$F(X) \prec Q,$$

where the unknown X takes values in a real vector space $\mathcal{X}$ (e.g., the space of symmetric matrices), the mapping $F : \mathcal{X} \to \mathbb{H}^n$ is linear, with $\mathbb{H}^n$ the set of Hermitian matrices, and $Q \in \mathbb{H}^n$. Despite the fact that we can already determine the stability of A from looking at its eigenvalues, (7.6) is of useful, for example when several system properties must be satisfied by or enforced on the system simultaneously through more complex LMIs, in addition to stability. The following theorem summarizes a number of useful facts for linear CT systems.

Theorem 7.1.3. *For the linear dynamical system (7.5), the following are equivalent*

1. *the zero solution is globally asymptotically stable.*
2. *the zero solution is globally exponentially stable.*
3. *For any $Q \succ 0$, the CT Lyapunov equation (7.7) has a unique solution $P \succ 0$.*
4. *For some $Q \succ 0$, the CT Lyapunov equation (7.7) has a unique solution $P \succ 0$.*
5. *The CT Lyapunov inequality (7.6) has a feasible solution $P \succ 0$.*

Remark 7.1.1. In fact we can further relax the conditions of the theorem by taking $Q = C^T C \succeq 0$ and (A, C) observable.

Here are some additional remarks about Lyapunov equations and inequalities.

Theorem 7.1.4. *Suppose A and Q are square matrices, with A Hurwitz. Then*

$$X = \int_0^\infty e^{A^* \tau} Q e^{A\tau} d\tau$$

is the unique solution to the Lyapunov equation $A^*X + XQ + Q = 0$.

Theorem 7.1.5. *Suppose $Q \succ 0$. Then A is Hurwitz if and only if there exists a solution $X > 0$ to the Lyapunov equation $A^*X + XA + Q = 0$. Equivalently, the matrix A is Hurwitz if and only if there exists $X > 0$ satisfying $A^*X + XA < 0$.*

Suppose A is Hurwitz. Then by scaling the left hand side of the Lyapunov inequality (7.6), we see that for any $Q \succ 0$, the Lyapunov inequalities

$$A^*X + XA + Q \prec 0,$$

and

$$A^*X + XA + Q \preceq 0,$$

have solutions. It turns out that for a fixed Q, the solution to the Lyapunov equation is the minimal solution of the Lyapunov inequality, as described in the following theorem.

Proposition 7.1.6. *Suppose that A is Hurwitz, and X_0 satisfies $A^*X_0 + X_0 A + Q = 0$, where Q is a symmetric matrix. If X satisfies $A^*X + XA + Q \preceq 0$, then*

$$X \succeq X_0.$$

Proof. We have immediately

$$-R := A^*(X - X_0) + (X - X_0)A \preceq 0.$$

Hence, by theorem 7.1.4, we have

$$X - X_0 = \int_0^\infty e^{A^* \tau} R e^{A\tau} d\tau \succeq 0,$$

i.e., $X \succeq X_0$. □

7.2 Discrete-Time Lyapunov Stability Theory

Consider the time-invariant discrete-time dynamical system

$$x_{k+1} = f(x_k),\ k \in \mathbb{N}, x_0 \text{ given}, \tag{7.8}$$

with f a continuous function $\mathcal{D} \to \mathcal{D}$, for some open set $\mathcal{D} \subset \mathbb{R}^n$. An equilibrium point of (7.8) is a point $x \in \mathcal{D}$ satisfying $f(x) = x$. For simplicity of notation we assume that 0 is an equilibrium point, i.e., $0 \in \mathcal{D}$ and $f(0) = 0$, and discuss stability notions for the zero solution $x_k \equiv 0$ of the discrete-time system (7.8).

Definition 7.2.1. Assume that $f(0) = 0$. Then the zero equilibrium point is

1. *Lyapunov stable* if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|x_0| < \delta$, then $|x_k| < \epsilon$, for all $k \in \mathbb{N}$.
2. *unstable* if it is not Lyapunov stable.
3. *asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $|x_0| < \delta$, then $\lim_{k\to\infty} x_k = 0$.
4. *geometrically stable* if there exist positive constants $\beta < 1$, α and δ such that if $|x_0| < \delta$, then $|x_k| \le \alpha |x_0| \beta^k, k \in \mathbb{N}$.
5. *globally asymptotically stable* if it is Lyapunov stable and for all $x_0 \in \mathbb{R}^n$, $\lim_{k\to\infty} x_k = 0$.
6. *globally geometrically stable* if there exist positive constants $\beta < 1$, α such that $|x_k| \le \alpha |x_0| \beta^k, k \in \mathbb{N}$, for all $x_0 \in \mathbb{R}^n$.

Sufficient conditions to show the various types of stability introduced in this definition are provided by Lyapunov's direct method, which works similarly to the continuous-time case. See, e.g., [HC08, chapter 13].

Theorem 7.2.1 (Lyapunov's direct method, discrete-time)**.** *Consider the discrete-time system (7.8), with $f(0) = 0$, and assume that there exists a continuous function $V : \mathcal{D} \to \mathbb{R}$ such that*

$$\begin{aligned} &V(0) = 0 \\ &V(x) > 0, \forall x \in \mathcal{D}, x \neq 0, \\ &V(f(x)) - V(x) \le 0, \forall x \in \mathcal{D}. \end{aligned} \tag{7.9}$$

Then the zero equilibrium point is Lyapunov stable. If (7.9) is replaced by the stronger inequality

$$V(f(x)) - V(x) < 0, \forall x \in \mathcal{D}, x \neq 0, \tag{7.10}$$

then the zero equilibrium point is asymptotically stable. If moreover $\mathcal{D} = \mathbb{R}^n$ and V is radially unbounded, i.e.,

$$V(x) \to \infty \ \text{ as } \|x\| \to \infty,$$

then (7.10) implies that the zero equilibrium point is globally asymptotically stable.

If there positive exist scalars α, β, $\rho < 1$, and $p \ge 1$ such that V satisfies

$$\alpha |x|^p \le V(x) \le \beta |x|^p, \quad \forall x \in \mathcal{D}, \tag{7.11}$$

$$V(f(x)) \le \rho V(x), \quad \forall x \in \mathcal{D}, \tag{7.12}$$

then the zero equilibrium point is geometrically stable (globably geometrically stable if $\mathcal{D} = \mathbb{R}^n$), and in fact

$$|x_k| \le \left(\frac{\beta}{\alpha}\right)^{1/p} |x_0| (\rho^{1/p})^k, \quad \forall k \in \mathbb{N}.$$

7.2.1 Application to Linear Systems and Quadratic Stability

Consider a DT linear system

$$x_{k+1} = Ax_k, \; k \in \mathbb{N}, \; x_0 \text{ given.}$$

It is well known that 0 is an asymptotically stable equilibrium of this system if and only if A is Schur, i.e., its spectral radius $\rho(A) := \max\{|\lambda| \text{ s.t. } \lambda \text{ is an eigenvalue of } A\}$ is strictly less than 1. In other words, all eigenvalues of A should be strictly within the unit circle. In this case, 0 is in fact geometrically stable (also called exponentially, as in continuous time), and trajectories are roughly bounded by ρ^k. Stability of this system turns out to be equivalent of quadratic stability, i.e., the existence of a quadratic Lyapunov function $V(x) = x^T Px$, with $P \succ 0$. Writing the condition 7.10, we obtain the condition

$$A^T PA - P \prec 0, \tag{7.13}$$

called a discrete-time Lyapunov inequality. To obtain geometric stability with an explicit bound on the convergence rate, we can use (7.12) which translates into

$$A^T PA - \rho P \prec 0, \;\; \rho < 1.$$

We can find a matrix P satisfying these conditions by solving a DT Lyapunov equation, i.e., for the form

$$A^T PA - P = -Q, \text{ where } Q \succ 0,$$

for example with $Q = I_n$. See the MATLAB command `dlyap`. Alternatively, these inequalities are *linear matric inequalities* (LMIs) and matrices P satisfying them can be found directly using semi-definite programming. This becomes particularly useful in more complicated design problems where guaranteeing stability is just one aspect of the system design problem, and additional specifications can be handled by LMIs as well.

7.3 Input-Output Methods for Stability Analysis

In this section, we consider sufficient conditions under which the L^p-stability of feedback connections of input-output systems can be guaranteed. The standard feedback configuration is shown on Fig. 7.1. The signals u_1, u_2 are the inputs, and the signals y_1, y_2, e_1, e_2 can be considered as outputs. Note that a slightly delicate point raised when considering such feedback configurations is the question of *well-posedness*. We say that the system is well-posed if for any $u_1, u_2 \in L^{pe}$, there are unique signals $e_1, e_2, y_1, y_2 \in L^{pe}$ satisfying the loop equations, i.e.,

$$u_1 = e_1 + H_2 e_2, \;\; y_1 = H_1 e_1 \tag{7.14}$$
$$u_2 = e_2 - H_1 e_1, \;\; y_2 = H_2 e_2. \tag{7.15}$$

The question of well-posedness is treated in references discussing input-output modeling, such as [DV09].

7.3.1 The Small-Gain Theorem

The small-gain theorem, in its various forms, is an apparently simple result which turns out to have far-reaching consequences and applications. For example, it is a basic tool in robust control. Roughly, small-gain theorems state that a feedback interconnection if stable if the "loop gain" is less than one.

Theorem 7.3.1 (small-gain theorem)**.** *Consider the system shown on Fig. 7.1, with $H_1, H_2 : L^{pe} \to L^{pe}$. Let $e_1, e_2 \in L^{pe}$, and* define *u_1, u_2 by (7.14), (7.15). Moreover suppose that there are constants $\gamma_1, \gamma_2, \beta_1, \beta_2$ such that*

$$\|(H_1 e_1)_T\|_p \leq \gamma_1 \|e_{1,T}\|_p + \beta_1, \;\; \forall T \geq 0, \tag{7.16}$$
$$\|(H_2 e_2)_T\|_p \leq \gamma_2 \|e_{2,T}\|_p + \beta_2, \;\; \forall T \geq 0. \tag{7.17}$$

Under these conditions, if $\gamma_1 \gamma_2 < 1$, then

$$\|e_{1,T}\|_p \leq (1 - \gamma_1\gamma_2)^{-1}(\|u_{1,T}\|_p + \gamma_2 \|u_{2,T}\|_p + \beta_2 + \gamma_2 \beta_1), \;\; \forall T \geq 0,$$
$$\|e_{2,T}\|_p \leq (1 - \gamma_1\gamma_2)^{-1}(\|u_{2,T}\|_p + \gamma_1 \|u_{1,T}\|_p + \beta_1 + \gamma_1 \beta_2), \;\; \forall T \geq 0.$$

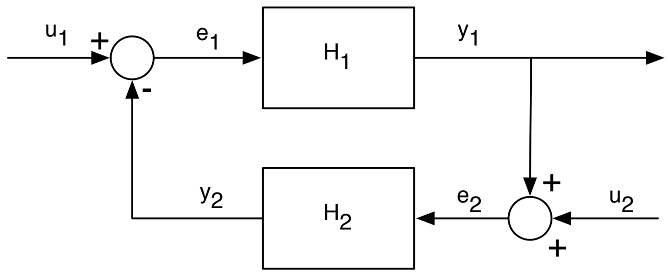

Figure 7.1: Standard Feedback Configuration

In particular if the system (7.14), (7.15) is well-posed, if H_1 and H_2 have finite L^p-gain γ_1, γ_2 and if $\gamma_1\gamma_2 < 1$, then the closed loop system system is finite-gain L^p-stable (i.e., as input-output system from (u_1, u_2) to (e_1, e_2) and hence also from (u_1, u_2) to (y_1, y_2)).

Proof. From (7.14), we have for all $T \geq 0$,

$$\begin{aligned}\|e_{1,T}\|_p &= \|u_{1,T} - (H_2 e_2)_T\|_p \\ &\leq \|u_{1,T}\|_p + \gamma_2 \|e_{2,T}\|_p + \beta_2.\end{aligned}$$

Similarly using (7.15)

$$\|e_{2,T}\|_p \leq \|u_{2,T}\|_p + \gamma_1 \|e_{1,T}\|_p + \beta_1.$$

Combining the two inequalities

$$\|e_{1,T}\|_p \leq \gamma_1\gamma_2 \|e_{1,T}\|_p + (\|u_{1,T}\|_p + \gamma_2 \|u_{2,T}\|_p + \gamma_2\beta_1 + \beta_2),$$

and the result follows. □

Remark 7.3.1. The theorem extends to relations H_1, H_2, i.e., multivalued maps. In this case, (7.16) and (7.16) must hold for all the possible outputs $H_i e_i$ of e_i, $i = 1, 2$. This fact is useful for example for feedback systems with hysteresis.

Note that in the second, most useful part of theorem (7.3.2), well-posedness is part of the assumptions. There are various sufficient conditions that guarantee well-posedness of a closed-loop system [DV09]. In addition, the incremental version of the small-gain theorem below does not require this assumption a priori.

Theorem 7.3.2 (incremental small-gain theorem)**.** *Suppose that both $H_1, H_2 : L^{pe} \to L^{pe}$ are incrementally finite-gain L^p-stable with incremental gains γ_1, γ_2 respectively. Then if $\gamma_1\gamma_2 < 1$, the closed loop system (7.14), (7.15) is well-posed and incrementally finite-gain L^p-stable (i.e., from (u_1, u_2) to (e_1, e_2) and also from (u_1, u_2) to (y_1, y_2)).*

Proof. Fix u_1, u_2, and note that we have unique solutions y_1, y_2 if and only if there are unique solutions e_1, e_2. We have

$$e_1 = u_1 - H_2(u_2 + H_1 e_1) =: F e_1. \tag{7.18}$$

Now note that

$$\begin{aligned}\|Fe_{1,T} - F\tilde{e}_{1,T}\|_p &= \|H_2(u_{2,T} + H_1 e_{1,T}) - H_2(u_{2,T} + H_1 \tilde{e}_{1,T})\|_p \\ &\leq \gamma_2 \|H_1 e_{1,T} - H_1 \tilde{e}_{1,T}\|_p \\ &\leq \gamma_1\gamma_2 \|e_{1,T} - \tilde{e}_{1,T}\|_p.\end{aligned}$$

Since by hypothesis $\gamma_1\gamma_2 < 1$, the map F is a contraction mapping, so that there is a unique solution $e_{1,T}$ and similarly $e_{2,T}$, given $u_{1,T}, u_{2,T}$.

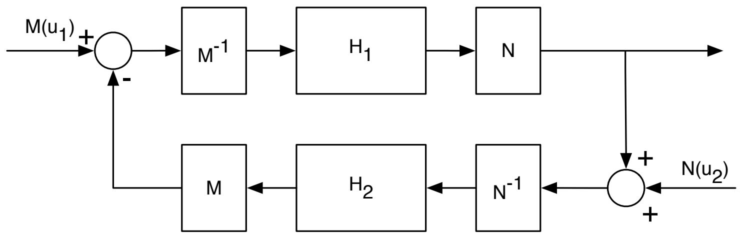

Figure 7.2: Introduction of Multipliers.

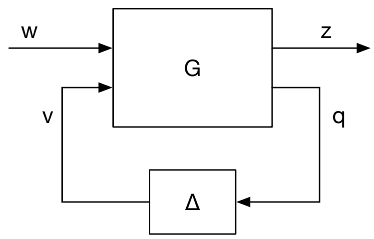

Figure 7.3: Set-up for robustness analysis.

Next for the incrementally finite gain stability of the closed-loop system, let u_1, u_2 and $\tilde{u}_1, \tilde{u}_2$ be two sets of inputs for the system, with corresponding outputs e_1, e_2 and $\tilde{e}_1, \tilde{e}_2$. Then from (7.18) we have

$$\|e_{1,T} - \tilde{e}_{1,T}\|_p \leq \|u_{1,T} - \tilde{u}_{1,T}\|_p + \gamma_2 \|u_{2,T} - \tilde{u}_{2,T}\|_p + \gamma_1\gamma_2 \|e_{1,T} - \tilde{e}_{1,T}\|_p,$$

so that

$$\|e_{1,T} - \tilde{e}_{1,T}\|_p \leq \frac{\|u_{1,T} - \tilde{u}_{1,T}\|_p + \gamma_2 \|u_{2,T} - \tilde{u}_{2,T}\|_p}{1 - \gamma_1\gamma_2}.$$

Similarly bounds hold for $\|e_2 - \tilde{e}_2\|_p, \|y_1 - \tilde{y}_1\|_p, \|y_2 - \tilde{y}_2\|_p$. □

Extending the Applicability of the Small Gain Theorem

In general, the small gain theorem is used in together with various transformations that reduce its conservativeness. A popular technique is to use multipliers, as shown on Fig. 7.2. The mapping M, N are assumed to have inverses M^{-1}, N^{-1}, and these four systems are assumed to be L^q-stable mappings. The new feedback system is essentially equivalent to the original one of Fig. 7.1. However, the small gain conditions now reads $\gamma(NH_1M^{-1})\gamma(MH_2N^{-1}) < 1$, for any choice of multipliers M, N. Finding the right multipliers can lead to a much tighter stability condition.

Another technique is to use loop transformations [Vid02].

Small-Gain Theorem for Robustness Analysis

The small gain theorem, together with multipliers, is heavily used in robust control to simultaneously certify the stability of an ensemble of dynamical systems, to which a particular system under study is assumed to belong. The overall configuration is show on Fig. 7.3. Typically, the system G is an LTI system, for which that gain γ_G can be computed precisely, and Δ is an unknown perturbation element, for which we have a gain bound.

Theorem 7.3.3. *Consider the configuration of Fig. 7.3, with G and Δ two L^p-stable systems with finite L^p-gains γ_G (from (v, w) to (q, z) and γ_Δ (from q to v). If $\gamma_G\gamma_\Delta < 1$, then the interconnection is L^p-stable (from w to z) with finite-gain at most γ_G.*

Proof. We have by definition, for any signals v, w, z, q,

$$\inf_T \{\gamma_G^p(\|v_T\|_p^p + \|w_T\|_p^p) - \|z_T\|_p^p - \|q_T\|_p^p\} > -\infty$$

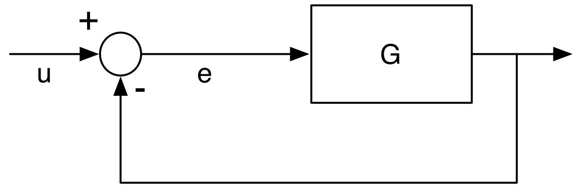

Figure 7.4: Unity Feedback System

and

$$\inf_T \{\gamma_\Delta^p (\|q_T\|_p^p - \|v_T\|_p^p)\} > -\infty.$$

Multiplying the second inequality by γ_G^p and summing them, we get

$$\inf_T \{\gamma_G^p \|w_T\|_p^p - \|z_T\|_p^p - (1 - (\gamma_\Delta \gamma_G)^p)\|q_T\|_p^p\} > -\infty.$$

When $(1 - (\gamma_\Delta \gamma_G)^p) > 0$, i.e., when $\gamma_\Delta \gamma_G < 1$, this gives

$$\inf_T \{\gamma_G^p \|w_T\|_p^p - \|z_T\|_p^p\} > -\infty.$$

□

Again, this theorem is generally used in combination with scalings or more general (possibly dynamic) multipliers.

Example 7.3.1 (A related example: robustness of feedback stability)**.** Consider the unity feedback system shown on Fig. 7.4. We have $e = u - Ge$, hence the input-output relation between u and e is given by $e = Hu = (I + G)^{-1}u$. Now let us assume that for this nominal closed-loop system, we have a gain condition with zero bias of the form

$$\|Hu\|_p \leq \gamma \|u\|_p. \tag{7.19}$$

We now want to study a perturbed version of the system where G is replaced by $G + \Delta$, for some uncertain operator Δ. We first express the perturbed operator in terms of the original one

$$\begin{aligned}
\tilde{H} &= [I + G + \Delta]^{-1} \\
&= [(I + \Delta(I + G)^{-1})(I + G)]^{-1} \\
&= (I + G)^{-1}[I + \Delta(I + G)^{-1}]^{-1} \\
&= H[I + \Delta H]^{-1}.
\end{aligned}$$

Now let M be any operator, with gain $\gamma(M)$ and zero bias $\beta_M = 0$. We can get a bound on the gain of $(I + M)^{-1}$ as follows.

$$(I + M)u = y \Rightarrow u = y - Mu \Rightarrow \|u\|_p \leq \|y\|_p + \gamma(M)\|u\|_p,$$

hence $\gamma((I + M)^{-1}) \leq 1/(1 - \gamma(M))$.

Coming back to our problem, assume that we have a rough characterization of the perturbation Δ in terms of its gain, and that we know

$$\gamma(\Delta) < \frac{1}{\gamma}.$$

Then $\gamma(\tilde{H}) \leq \gamma/(1 - \gamma\gamma(\Delta))$, and so the perturbed system is L^p finite gain stable.