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Dissipative Dynamical Systems

The notion of dissipativity is of fundamental theoretical and practical importance in control, and was introduced and studied in particular in the early work of J. C. Willems [Wil71b, Wil72a, Wil72b]. In a sense, it establishes a natural link between the properties of input-output and state-space models. It also forms the foundation of many modern computational tools for the analysis and synthesis of control systems based on solving Linear Matrix Inequalities (LMIs), which have received considerable attention for the past two decades. Some references on dissipativity include [van17, SW05] ([SW05] does not define a storage function to be necessarily nonnegative, however).

4.1 Dissipative State-Space Systems

4.1.1 The Dissipation Inequality

Consider a continuous-time, time-invariant dynamical system Σ with input $u(t) \in \mathsf{U}$, output $y(t) \in \mathsf{Y}$, state $x(t) \in \mathsf{X}$ and state-space representation

$$\dot{x} = f(x, u) \tag{4.1}$$

$$y = g(x, u). \tag{4.2}$$

We assume that at least for a certain class of signals $\mathcal{U}$, a solution to (4.1), (4.2) for a given initial condition exists and is unique. Moreover, recall that the input-output relation between u and y defined by such a state-space system is necessarily causal.

Let $\sigma : \mathsf{U} \times \mathsf{Y} \to \mathbb{R}$ be a mapping such that $t \to \sigma(u(t), y(t))$ is locally absolutely integrable for all input-output pairs (u, y) satisfying (4.1), (4.2), i.e., $\int_{t_0}^{t_1} |\sigma(u(t), y(t))| dt < \infty$, for all $t_0, t_1 \in \mathbb{R}$. The mapping σ is called a *supply function* or *supply rate*.

Definition 4.1.1 (dissipativity)**.** The system Σ with supply rate σ is said to be *dissipative* with respect to the supply rate σ if there exists a nonnegative function $V : \mathsf{X} \to \mathbb{R}_+$, called a *storage function*, such that

$$V(x(t_1)) \leq V(x(t_0)) + \int_{t_0}^{t_1} \sigma(u(t), y(t)) dt \tag{4.3}$$

for all $t_0 \leq t_1$ and all signals (u, x, y) satisfying (4.1), (4.2). Σ is said to be *conservative* or *lossless* with respect to σ if equality holds in (4.3) for all $t_0 \leq t_1$ and all signals (u, x, y) satisfying (4.1), (4.2). Finally, it is said to be *cyclo-dissipative* if $V(x)$ is not necessarily nonnegative for all x.

The storage function generalizes the notion of an energy function. The supply rate $\sigma(u(\cdot), y(\cdot))$ can be interpreted as the rate at which energy flows *into* the system if the system generates the input-output pair (u, y). So on a time interval $[0, T]$, work has been done *on* the system if $\int_0^T \sigma(u(t), y(t)) dt \geq 0$ is positive, and is done *by* the system otherwise. Inequality (4.3) is called the *dissipation inequality*. Without interpretation, it means that the sum of the $V(x(t_0))$ initially stored in the system and the energy $\int_{t_0}^{t_1} \sigma(u(t), y(t)) dt$ supplied to the system during $[t_0, t_1]$ is greater or equal to the final energy $V(x(t_1))$ stored in the system. In other words, the system has not created energy during (any interval) $[t_0, t_1]$, and has dissipated a strictly positive amount of energy during that interval if the inequality is strict. Part of the energy supplied is stored, and part is dissipated.

Note that when $t \to V(x(t))$ is differentiable, then the dissipation inequality (4.3) is equivalent to

$$\frac{d}{dt}V(x(t)) = \left[\frac{\partial V}{\partial x}|_{(x(t))}\right] \cdot f(x(t), u(t)) \leq \sigma(u(t), y(t))$$

for all t and all solutions $(u(\cdot), x(\cdot), y(\cdot))$ of (4.1), (4.2). Hence, Σ is dissipative with respect to σ if

$$\left[\frac{\partial V}{\partial x}|_x\right] \cdot f(x, u) \leq \sigma(u, g(x, u)) \tag{4.4}$$

holds for all points $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Here $\frac{\partial V}{\partial x}|_x$ denotes the row vector of partial derivatives $\partial V/\partial x_i$, for $i = 1, \ldots, n$, evaluated at x. We call (4.4) the *differential dissipation inequality.* It states that the rate of change of storage along trajectories of the system never exceeds the rate of supply.

Remark 4.1.1. A system can be simultaneously dissipative with respect to several supply functions. Each additional dissipation inequality further constrains the trajectories of the system. For example, a thermodynamic system at uniform temperature T on which mechanical work is being done at rate W and which is being heated at rate Q is lossless with respect to $\sigma_1 := W + Q$ with storage function the internal energy and dissipative with respect to $-Q/T$ with storage function the entropy [SW05, Chapter 2].

4.1.2 Quadratic Supply Rates

In Section 4.2 we will come back to the fundamental questions of certifying when a system is dissipative with respect to a given supply rate, i.e., when a storage function exists. First however, let us consider the consequences of dissipativity for the special case of quadratic supply rates, which is also the case most amenable to computations. A particularly important class of supply functions $\sigma : \mathsf{U} \times \mathsf{Y} \to \mathbb{R}$ are the quadratic forms

$$\sigma(u, y) = \begin{bmatrix} u \\ y \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = u^T Q u + 2u^T S y + y^T R y. \tag{4.5}$$

They arise in network theory, bond graph theory, scattering theory, H^∞ theory, game theory and linear quadratic and H^2 optimal control theory, among others. Among them, two are even more important. The first one is

$$\sigma(u, y) = \gamma^2 |u|^2 - |y|^2, \tag{4.6}$$

i.e., $Q = \gamma^2 I$, $R = -I$ and $S = 0$. This reminds us of the notion of L^2-gain. Indeed, we have the following proposition.

Proposition 4.1.1. *Suppose that Σ defined by (4.1), (4.2) is dissipative with respect to the supply rate $\sigma(u, y) = \gamma^p |u|^p - |y|^p$, for some $p \geq 1$ and some $\gamma > 0$. Then, as an input-output system, $\Sigma \subset L^{pe} \times L^{pe}$ has L^p-gain at most γ.*

Proof. Note that each initial condition x_0 defines a different input-output map $u(\cdot) \to y(\cdot)$ by (4.1), (4.2). We have by dissipativity et nonnegativity of V that for any $(u, y) \in \Sigma$,

$$0 \leq V(x(T)) \leq V(x_0) + \int_0^T \gamma^p |u|^p - |y|^p dt, \forall T \geq 0$$

$$\text{hence, } \|y_T\|_p^p \leq \gamma^p \|u_T\|_p^p + V(x_0),$$

which is the definition of finite L^p-gain bounded by γ. ☐

The second fundamental example of supply rate is $\sigma(u, y) = u^T y$, i.e., u and y have the same dimensions, $Q = R = 0$ and $S = I/2$ in (4.5). In this case, the system Σ (as input-output system or state-space system) is called passive, a name motivated by the theory of electrical circuits. We will come back to this case if needed.

The differential dissipation inequality leads to computational methods to compute the L^2-gain. For example, suppose Σ is of the input-affine form

$$\begin{aligned}\dot{x} &= f(x) + G(x)u \\ y &= h(x),\end{aligned}$$

where $G(x)$ is an $n \times m$ matrix for all x. Then (4.4) with (half of the) supply rate (4.6) reads

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2}|h(x)|^2 \leq \frac{1}{2}\gamma^2 |u|^2 - \frac{\partial V}{\partial x} G(x)u,$$

which needs to be valid for all x and all u, in particular for the u minimizing the right-hand side. Carrying out this simple quadratic minimization, we obtain

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x} G(x)G(x)^T \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h(x)^T h(x) \leq 0, \forall x \in \mathbb{R}^n.$$

This is called a Hamilton-Jacobi inequality, a partial differential inequality in the unknown function V. If it has a solution, then the L^2-gain of the system is less than γ. A particular case of interest is the linear case

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du,\end{aligned}$$

where the term Du has been added for generality. In this case, searching for a quadratic storage function of the form $V(x) = x^T P x$ for $P \succeq 0$, the procedure above leads to the quadratic inequality

$$A^T P + PA + C^T C + (PB + C^T D)(\gamma^2 I - D^T D)^{-1}(PB + C^T D)^T \preceq 0,$$

provided $\gamma > \sigma_{max}(D)$, which guarantees the positive definiteness of $\gamma^2 I - D^T D$. By the Schur complement, this inequality is equivalent to the Linear Matrix Inequality (LMI)

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} \preceq 0,$$

which can in fact be found directly more simply by writing the dissipation inequality as a quadratic form in both x and u. This LMI can take various equivalent forms found in the literature. For example, one can show by another Schur complement argument that is is equivalent to

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \preceq 0.$$

To arrive at these LMIs, we have made the initial assumption that a quadratic storage function always exists for the LTI system and supply rate (4.6) when the L^2 gain is bounded by γ. In fact, a deep result says that indeed quadratic storage functions are always sufficient to show dissipativity of LTI systems with general quadratic supply rates. For the special case of the L^2 gain above, this result is called the *bounded real lemma*, and for the case of passivity, it is called the *positive real lemma*.

Remark 4.1.2. For completeness, the LMI to certify dissipativity of an LTI system with respect to (4.5) (based on a storage function $x^T P x$) takes the form $P \succeq 0$, $F(P) \preceq 0$, with the following equivalent forms

$$\begin{aligned} F(P) &:= \begin{bmatrix} I & A^T \\ 0 & B^T \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} - \begin{bmatrix} 0 & C^T \\ I & D^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} A^T P + PA - C^T RC & PB - C^T S - C^T RD \\ B^T P - SC - D^T RC & -Q - SD - D^T S^T - D^T RD \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & -Q & -S \\ 0 & 0 & -S^T & -R \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix}. \end{aligned}$$

Using the Schur complement and denoting $W := Q + SD + D^T S^T + D^T RD$, the LMI $F(P) \prec 0$ is also equivalent to $W \succ 0$ together with the quadratic inequality in P

$$A^T P + PA - C^T RC + (PB - C^T S^T - C^T RD)^{-1} W^{-1} (B^T P - SC - D^T RC) \prec 0.$$

This is typically used in the other direction, with the Schur complement allowing us to convert the quadratic inequality into an LMI that can be handled by semi-definite programming solvers.