A Dynamic Game Model of Collective Choice in Multi-Agent Systems

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***Abstract*— Inspired by socially influenced decision making mechanisms such as the collective behavior of cancer cells, honey bees searching for a new colony or the mobility of bacterial swarms, we consider a Mean Field Games (MFG) model of collective choice where a large number of agents choose between multiple alternatives while taking into account the group's behavior. For example, in elections, individual interests and collective opinion swings together contribute in the crystallization of final decisions. At first, the agents' decisions are determined by their initial states. Subsequently, the model is generalized to include a priori individual preferences towards the destination points. For example, personal preferences that transcend party lines in elections. We show that multiple strategies exist with each one of them defining an epsilon Nash equilibrium.**

I. Introduction

Collective decision making occurs when a large number of agents choose between multiple alternatives while influenced by the group's behavior. This phenomenon is abundant in most social structures, e.g., in socio-economic systems [1], biological populations [2], or human societies. Socially influenced decision making is characterized by the weak coupling of the agents, i.e., (i) they are considerably influenced by the (anonymous) group's behavior, (ii) their isolated individual behaviors have negligible impact on the population's attitude and (iii) they collectively reproduce the global behavior of the group. For example, in elections, an individual-social trade off occurs, where individual interests and collective opinion swings together contribute in the crystallization of final decisions [3]–[5].

A related topic in economics is discrete choice models where an agent chooses between multiple alternatives such as the mode of transportation [6], entry and withdrawal from the labor market, or residential location [7]. In many circumstances, these individual choices are influenced by the so called "Peer Effect", "Neighbourhood Effect" or "Social Effect". For example, smoking decision in schools [8]. Brock and Durlauf propose in [9] a static binary discrete choice model of a large population, which takes into account the social effect as represented by the mean of the population. They use an approach similar to MFG to show that, for an infinite size population, each agent can predict the mean of the population by a fixed point calculation, and make a decentralized choice based upon its prediction. Moreover, they show that multiple anticipated means may exist. Our analysis leads to similar insights for a dynamic noncooperative multiple choice game, including situations where the agents have limited information about the dynamics of other agents.

In this paper, we consider situations where a large number of agents must move, within a finite time horizon, from their initial positions towards one of multiple destination points. Along the path, they must remain grouped and develop as little effort as possible. A related topic, in stochastic optimal control, is the "Homing" problem which was introduced first by Whittle and Gait in [10] and studied later in [11], [12] for example. However, this problem is concerned with situations where a single agent tries to reach one of multiple predefined targets.

Our purpose is to model situations where individual decisions both depend on and collectively influence the group's behavior. In [13], we considered the binary choice case where each agent knows the exact initial conditions of the other players. In this paper, we consider a model with multiple destination points. Moreover, we assume that each agent has limited information about the other agents' initial states in the form of a common initial probability distribution. This would represent, for example, the individual opinions privacy in the elections example. In addition to the randomized limited information about the initial states, we generalize the problem to consider nonuniform agents with initial preferences towards the final destinations. For example, even though influenced by the exogenous peers' smoking behavior, a teenager's decision to smoke depends on a priori affinities towards the final decision "smoking" or "non smoking". These affinities are the result of many endogenous factors, such as education, parental pressure, financial condition, health, cigarette's taste etc. In the elections example this would reflect personal preferences that transcend party lines for example.

A. Mathematical Model

We consider a noncooperative dynamic game, involving N agents, with identical linear dynamics

$$\dot{x}_i = Ax_i + Bu_i \qquad \forall i \in \{1\ldots,N\}, \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state of agent i and $u_i \in \mathbb{R}^m$ its control input. We assume that each agent has a limited information about the other agents initial conditions in the form of a statistical distribution. Thus, we assume that the initial conditions x_i^0, $i = 1,\ldots,N$, are independent and identically distributed (i.i.d.) on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Each

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agent i is associated with an individual cost function

$$J_i(u_i, \bar{x}, x_i^0) = \mathbb{E}^{x^0_{1:N}}\left(\int_0^T \left\{ \frac{q}{2}\|x_i - \bar{x}\|^2 + \frac{r}{2}\|u_i\|^2 \right\} \mathrm{d}t + \frac{M}{2} \min_{j=1,\dots,l} \left(\|x_i(T) - p_j\|^2 \right) \middle| x_i^0 \right) \quad (2)$$

where $\bar{x} = 1/N \sum_{j=1}^N x_j,\ q, r$ are positive constants and M is a large positive number. These costs force the agents to stay grouped around the mean, to expend as little effort as possible and to reach, before a time T, one of the predefined destinations p_j, $j = 1, \dots, l$. We seek an $\epsilon-$Nash equilibrium [14], a sort of Nash equilibrium approximation, where ϵ converges to zero as the size of the population goes to infinity.

Definition 1: Consider N players, a set of strategy profiles $S = S_1 \times \cdots \times S_N$ and for each player k a payoff function $J_k(u_1, \dots, u_N)$, $\forall (u_1, \dots, u_N) \in S$. A strategy $(u_1^*, \dots, u_N^*) \in S$ is called an $\epsilon-$Nash equilibrium with respect to the costs J_k, if there exists an $\epsilon > 0$ such that for any fixed $1 \le i \le N$, $\forall u_i \in S_i$, we have

$$J_i(u_1^*, \dots, u_{i-1}^*, u_i, , u_{i+1}^*, \dots, u_N^*) \geq J_i(u_1^*, \dots, u_N^*) - \epsilon.$$

Inspired by the MFG theory [14]–[17], we seek a set of decentralized strategies satisfying the $\epsilon-$Nash property. To compute its strategy, each agent will then need only to know its own state and the common initial distribution of the agents. The minimum term in the cost makes the problem non-standard with respect to the Mean Field Games (MFG) literature cited below. Unlike these standard papers, the proofs, given in this paper, for the existence of a fixed point macroscopic behavior are based on topological fixed point theorems, namely Brouwer's and Schauder's fixed point Theorems. This results in weak assumptions and multiplicity of solutions. The latter would explain the existence of multiple social behaviors in many similar situations. However, as it will be shown later, the computation of these fixed points requires more sophisticated methods than the simple iterations used in the case of contraction maps. As described in [14], [15], [17], the MFG approach assumes a continuum of agents to which one can ascribe an assumed given deterministic macroscopic behavior (the mean field), captured here by the mean trajectory $\bar{x}$ set equal to some posited yet unknown deterministic trajectory $\hat{\bar{x}}$. The cost functions being now decoupled, each agent optimally tracks $\hat{\bar{x}}$. The resulting control laws depend on the local states and $\hat{\bar{x}}$. The solution of the tracking problem is presented in Section II. By implementing the resulting decentralized strategies in their dynamics, the agents reproduce a new candidate tracking path obtained as the mean agent population trajectory. In line with MFG analysis, the posited tracked path is considered an acceptable candidate only if it is replicated by the mean of the agents when they optimally track it. Thus, we look for candidate trajectories that are fixed points of the tracking path to tracking path map defined above. The fixed points of the tracking path to tracking path map are studied in Section III. In Section IV, the problem is generalized to include initial preferences towards the destination points. In this case, the agents are assumed to be nonuniform and each agent has limited information about the other agents dynamic parameters, in the form of a statistical distribution over the matrices A and B. We show in Section V that the decentralized strategies developed when tracking the fixed point trajectories in all the cases considered above constitute $\epsilon-$Nash equilibria with ϵ going to zero as N goes to infinity. In Section VI we provide some numerical simulation results, while Section VII presents our conclusions.

B. Notation

The following notation is used throughout the paper. $C(X, Y)$ denotes the set of continuous functions from a normed vector space X to $Y \subset \mathbb{R}^k$ with the standard supremum norm $\|.\|_\infty$. $x_{i:j}$ denotes the vector $(x_i, x_{i+1}, \dots, x_j)$. $\mathbb{P}(A)$ denotes the probability of an event A. $\mathbb{E}^x$ denotes the expectation with respect to the mesure induced by the random variable x. 1_X denotes the indicator function of a subset X. The transpose of a matrix M is denoted by M^T. I_k refers to the identity $k \times k$ matrix. Subscript i is used to index entities related to the agents. Subscripts j and k are used to index entities related to the home destinations. $\{X \le x\}$ denotes the set $\{\omega \in \Omega | X(\omega) \le x\}$ where X is a random variable on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $x \in \mathbb{R}$.

Remark 1: For the proofs of theorems and lemmas one can refer to [18].

II. Tracking Problem and Basins of attraction

Following the MFG approach and the analysis in [13], we assume the trajectory $\bar{x}(t)$ in (2) to be given for now and equal to $\hat{\bar{x}}(t)$. The cost function (2) can be written

$$J_i(u_i, \hat{\bar{x}}, x_i^0) = \min_{j=1,\dots,l} J_{ij}(u_i, \hat{\bar{x}}, x_i^0), \quad (3)$$

where

$$J_{ij}(u_i, \hat{\bar{x}}, x_i^0) = \int_0^T \left\{ \frac{q}{2}\|x_i - \hat{\bar{x}}\|^2 + \frac{r}{2}\|u_i\|^2 \right\} \mathrm{d}t + \frac{M}{2}\|x_i(T) - p_j\|^2, \quad (4)$$

Moreover, we have

$$\inf_{u_i(.)} J_i(u_i, \hat{\bar{x}}, x_i^0) \quad = \quad \min_{j=1,\dots,l} \left(\inf_{u_i(.)} J_{ij}(u_i, \hat{\bar{x}}, x_i^0), \right).$$

Assuming a full (local) state feedback, the optimal control for (3) is

$$u_i^* = u_{ij}^* \quad \text{ if } J_{ij}(u_{ij}^*, \hat{\bar{x}}, x_i^0) = \min_{k=1,\dots,l} J_{ik}(u_{ik}^*, \hat{\bar{x}}, x_i^0)$$

where u_{ik}^* is the optimal solution of the simple linear quadratic tracking problem with cost function J_{ik}, namely [19]

$$u_{ik}^*(t) = -\frac{1}{r} B^T \Big(\Gamma(t) x_i + \beta_k(t) \Big), \quad \forall k \in \{1, \dots, l\}$$

with corresponding optimal costs

$$J_{ik}^*(\hat{\bar{x}}, x_i^0) = \frac{1}{2}(x_i^0)^T \Gamma(0) x_i^0 + \beta_k(0)^T x_i^0 + \delta_k(0),$$

and where Γ, β_k and δ_k are respectively matrix-, vector-, and real-valued functions, satisfying the following backward propagating differential equations:

$$\dot{\Gamma} - \frac{1}{r}\Gamma BB^T\Gamma + \Gamma A + A^T\Gamma + qI_n = 0 \tag{5a}$$

$$\dot{\beta}_k = \left(\frac{1}{r}\Gamma BB^T - A^T\right)\beta_k + q\hat{\bar{x}} \tag{5b}$$

$$\dot{\delta}_k = \frac{1}{2r}(\beta_k)^T BB^T\beta_k - \frac{1}{2}q\hat{\bar{x}}^T\hat{\bar{x}}, \tag{5c}$$

with the final conditions

$$\Gamma(T) = MI_n,\ \ \beta_k(T) = -Mp_k,\ \ \delta_k(T) = \frac{1}{2}Mp_k^T p_k.$$

We summarize the above analysis in the following lemma.

Lemma 1: The tracking problem (3) has a unique optimal control law

$$u_i^*(t) = -\frac{1}{r}B^T\big(\Gamma(t)x_i + \beta_j(t)\big) \quad \text{ if } x_i^0 \in D_j(\hat{\bar{x}}) \tag{6}$$

where Γ, β_j, δ_j are the unique solutions of (5a)-(5c), and

$$D_j(\hat{\bar{x}}) = \Big\{x \in \mathbb{R}^n \text{ such that}$$
$$\big(\beta_j(0) - \beta_k(0)\big)^T x \le \delta_k(0) - \delta_j(0),\ \forall k = 1,\dots,l\Big\}. \tag{7}$$

By solving (5b) and (5c), (7) can be written as follows:

$$D_j(\hat{\bar{x}}) = \Big\{x \in \mathbb{R}^n \text{ such that}$$
$$(\beta_{jk})^T x \le \delta_{jk}^{(1)} + \delta_{jk}^{(2)}(\hat{\bar{x}}),\ \forall k = 1,\dots,l\Big\} \tag{8}$$

where $\Pi(t) = \frac{1}{r}\Gamma(t)BB^T - A^T$, Φ_Π is the unique solution of

$$\frac{d\Phi_\Pi(t,T)}{dt} = \Pi\Phi_\Pi(t,T),\quad \Phi_\Pi(T,T) = I_n, \tag{9}$$

$\Psi(\eta_1,\eta_2,\eta_3,\eta_4) = \Phi_\Pi^T(\eta_1,\eta_2)BB^T\Phi_\Pi(\eta_3,\eta_4)$ and

$$\begin{aligned}
\beta_{jk} &= M\Phi_\Pi(0,T)(p_k - p_j)\\
\delta_{jk}^{(1)} &= \frac{1}{2}Mp_k^T p_k - \frac{1}{2}Mp_j^T p_j\\
&\quad + \frac{M^2}{2r}p_k^T\int_T^0 \Psi(\eta,T,\eta,T)\,\mathrm{d}\eta\, p_k\\
&\quad - \frac{M^2}{2r}p_j^T\int_T^0 \Psi(\eta,T,\eta,T)\,\mathrm{d}\eta\, p_j\\
\delta_{jk}^{(2)}(\hat{\bar{x}}) &= \frac{Mq}{r}(p_j - p_k)^T\int_T^0\int_T^\eta \Psi(\eta,T,\eta,\sigma)\hat{\bar{x}}(\sigma)\,\mathrm{d}\sigma\mathrm{d}\eta.
\end{aligned} \tag{10}$$

Given any continuous path $\hat{\bar{x}}(t)$, there exist l basins of attraction where all the agents initially in $D_j(\hat{\bar{x}})$ go towards p_j for $j = 1,\dots,l$. Therefore, the mean of the population is highly dependent on the structure of $D_j(\hat{\bar{x}})$, $j = 1\dots,l$.

Optimal control laws (6) depend on the tracked path $\hat{\bar{x}}(t)$ and the local state x_i. As mentioned above, each agent should reach one of the predefined destinations. The next lemma establishes that for any control horizon length T, M can be made large enough that each agent reaches an arbitrarily small neighborhood of some destination point by applying the control law (6).

Lemma 2: Suppose that the pair (A,B) is controllable and the agents are optimally tracking a path $\hat{\bar{x}}(t)$ that is replicated by the mean of the population. Then, for any $\epsilon > 0$ there exists an $M > 0$ such that each agent, at time T, is in a ball of radius ϵ and centered at one of the potential final destinations of the agents.

III. Fixed Point

Having presented the solution of the general tracking problem, we now look for candidate continuous paths $\hat{\bar{x}}(t)$ which are sustainable, in that they are replicated as means of the agent population under the associated optimal tracking policies. We start our search for a fixed point path by replacing $\bar{x}$ in (2) by a continuous path $\hat{\bar{x}}$. By Lemma 1, there exist l regions $D_j(\hat{\bar{x}})$ such that the agents initially in $D_j(\hat{\bar{x}})$ select the control law $-\frac{1}{r}B^T\Big(\Gamma x + \beta_j\Big)$ when tracking $\hat{\bar{x}}$. By substituting the solution of (5b) in (6) and the resulting control law in (1), we show that the mean of a generic agent with initial condition x_i^0 is equal to

$$\begin{aligned}
\bar{x}_\infty &= \mathbb{E}^{x_i^0}x_i(t) = \Phi_\Pi^T(0,t)\mu_0\\
&\quad + \sum_{j=1}^{l}\mathbb{P}\Big(x_i^0 \in D_j(\hat{\bar{x}})\Big)\frac{M}{r}\int_0^t \Psi(\sigma,t,\sigma,T)p_j\,\mathrm{d}\sigma\\
&\quad - \frac{q}{r}\int_0^t\int_T^\sigma \Psi(\sigma,t,\sigma,\tau)\hat{\bar{x}}(\tau)\,\mathrm{d}\tau\mathrm{d}\sigma
\end{aligned} \tag{11}$$

where μ_0 is the initial population mean. Equation (11) defines an operator G_∞ that maps the tracked path $\hat{\bar{x}}$ to the mean $\bar{x}_\infty$. G_∞ and its fixed points if any, only depend on the initial statistical distribution of the agents. By replacing the probabilities in (11) by an arbitrary $\lambda = (\lambda_1,\dots,\lambda_l) \in [0,1]^l$, we define a new map

$$\begin{aligned}
T_\lambda(\hat{\bar{x}}) &= \Phi_\Pi^T(0,t)\mu_0 + \sum_{j=1}^{l}\lambda_j\frac{M}{r}\int_0^t \Psi(\sigma,t,\sigma,T)p_j\,\mathrm{d}\sigma\\
&\quad - \frac{q}{r}\int_0^t\int_T^\sigma \Psi(\sigma,t,\sigma,\tau)\hat{\bar{x}}(\tau)\,\mathrm{d}\tau\mathrm{d}\sigma.
\end{aligned} \tag{12}$$

Lemma 3: Consider $\lambda = (\lambda_1,\dots,\lambda_l) \in [0,1]^l$. T_λ has a unique fixed point equal to

$$y_\lambda = R_1(t)\mu_0 + R_2(t)p_\lambda \tag{13}$$

where $p_\lambda = \sum_{j=1}^{l}\lambda_j p_j$, $\tilde{R}_1 = \Phi_\Pi^T(t,0)R_1(t)$ and $\tilde{R}_2 = \Phi_\Pi^T(t,0)R_2(t)$ are the unique solutions of

$$\begin{aligned}
\dot{\tilde{R}}_1 &= -\frac{q}{r}\int_T^t \Psi(t,0,t,\sigma)\Phi_\Pi^T(0,\sigma)\tilde{R}_1(\sigma)\,\mathrm{d}\sigma\\
\dot{\tilde{R}}_2 &= -\frac{q}{r}\int_T^t \Psi(t,0,t,\sigma)\Phi_\Pi^T(0,\sigma)\tilde{R}_2(\sigma)\,\mathrm{d}\sigma\\
&\quad + \frac{M}{r}\Psi(t,0,t,T)
\end{aligned} \tag{14}$$

with initial conditions $\tilde{R}_1(0) = I_n,\ \tilde{R}_2(0) = 0$.

We now define

$$\theta_{jk}^{(1)} = \frac{Mq}{r}(p_j^T - p_k^T)\int_T^0 \int_T^\eta \Psi(\eta,T,\eta,\sigma)R_1(\sigma)\,\mathrm{d}\sigma\mathrm{d}\eta$$
$$\theta_{jk}^{(2)} = \frac{Mq}{r}(p_j^T - p_k^T)\int_T^0 \int_T^\eta \Psi(\eta,T,\eta,\sigma)R_2(\sigma)\,\mathrm{d}\sigma\mathrm{d}\eta.$$

The next theorem establishes the existence of a fixed point of G_∞. We define F_∞ a map from $[0,1]^l$ into itself such that

$$F_\infty(\lambda_1,\dots,\lambda_l) = \begin{bmatrix} \mathbb{P}\Big((\beta_{1j})^T x_i^0 \le \delta_{1j}^{(1)} + \theta_{1j}^{(1)}\mu_0 + \theta_{1j}^{(2)} p_\lambda,\ \forall j=1,\dots,l\Big) \\ \vdots \\ \mathbb{P}\Big((\beta_{lj})^T x_i^0 \le \delta_{lj}^{(1)} + \theta_{lj}^{(1)}\mu_0 + \theta_{lj}^{(2)} p_\lambda,\ \forall j=1,\dots,l\Big) \end{bmatrix}^T.$$

Assumption 1 : We assume that P_0 is such that the P_0-measure of hyperplanes of dimension lower than n is zero.

Theorem 4: The following statements hold:

1) $\hat{\bar{x}}$ is a fixed point of G_∞ if and only if there exists $\lambda = (\lambda_1,\dots,\lambda_l)$ in $[0,1]^l$ such that
$$F_\infty(\lambda) = \lambda \tag{15}$$
for $\hat{\bar{x}}(t) = R_1(t)\mu_0 + R_2(t)p_\lambda$.
2) F_∞ has a fixed point (equivalently G_∞ has a fixed point).

In Theorem 4, the first point shows that computing the anticipated macroscopic behaviors is equivalent to computing all the λ's satisfying (15) under the corresponding constraint on $\hat{\bar{x}}(t)$. To compute the λ satisfying (15), each agent is assumed to know the initial statistical distribution of the agents. Multiple λ's may exist. Hence, an a priori agreement on how to choose λ should exist. In that respect, agents may implicitly assume that collectively they will opt for the λ (assuming it is unique!) that minimizes the total expected population cost

$$\mathbb{E}^{x_i^0} J_i\Big(u_i^*(x_i,\hat{\bar{x}}),\hat{\bar{x}}),x_i^0\Big) = \mathbb{E}^{x_i^0} \min_{k=1,\dots,l}\Big\{\frac{1}{2}(x_i^0)^T\Gamma(0)x_i^0 + \beta_k(0)^T x_i^0 + \delta_k(0)\Big\}.$$

The latter can be evaluated if the initial statistical distribution of the agents is a shared information.

A. Computation of The Fixed Points

The map F_∞ is not necessary a contraction. Hence, it is sometimes impossible to compute its fixed points by the simple iterative method $\lambda_{k+1} = F_\infty(\lambda_k)$. Moreover, the computation of its Jacobian matrix inverse is computationally expensive since F_∞ is a vector of probabilities of some regions delimited by hyperplanes. Hence, we use Broyden's method [20], a Quasi-Newton method, to find a solution for the nonlinear equation (15). Unlike Newton's method, this method updates the inverse of the Jacobian, at the root estimate, recursively. Having computed a fixed point λ, the agents can compute the corresponding fixed point of G_∞, $\hat{\bar{x}}$ which is equal to $R_1(t)\mu_0 + R_2(t)p_\lambda$.

IV. Nonuniform Population with Initial Preferences

In the previous section, the agents' final decisions are dictated by their initial positions. In the following, we generalize the model by considering that, in addition to their initial positions, the agents are affected by a priori opinion. Moreover, we assume in this section that the agents have nonuniform dynamics. We consider N agents with nonuniform dynamics

$$\dot{x}_i = A_i x_i + B_i u_i \qquad i=1,\dots,N, \tag{16}$$

We define the individual cost function

$$J_i(u_i,\bar{x},x_i^0) = \mathbb{E}^{x_{1:N}^0}\Bigg(\int_0^T \Big\{\frac{q}{2}\|x_i-\bar{x}\|^2 + \frac{r}{2}\|u_i\|^2\Big\}\,\mathrm{d}t + \min_{j=1,\dots,l}\Big(\frac{M_{ij}}{2}\|x_i(T)-p_j\|^2\Big)\Big|x_i^0\Bigg). \tag{17}$$

It is convenient, when considering the limiting population, to represent the limiting sequences of $(\theta_i)_{i=1,\dots,N} = ((A_i,B_i))_{i=1,\dots,N}$ and $(M_i)_{i=1,\dots,N} = ((M_{i1},\dots,M_{il}))_{i=1,\dots,N}$ by two independent random variables θ and M in compact sets Θ and W. Let us denote the empirical measures of the sequences θ_i and M_i, $P_\theta^N(\mathcal{A}) = 1/N\sum_{i=1}^N 1_{\{\theta_i\in\mathcal{A}\}}$ and $P_M^N(\mathcal{A}) = 1/N\sum_{i=1}^N 1_{\{M_i\in\mathcal{A}\}}$ for all (Borel) measurable sets $\mathcal{A}$. We assume that P_θ^N and P_M^N have weak limits P_θ and P_M. For further discussions about this assumption, one can refer to [21].

We develop the following analysis for a generic agent with an initial position x^0, dynamical parameters $\theta = (A_\theta,B_\theta)$ and initial preference vector $M = (M_1,\dots,M_l)$. Assuming an infinite size population, we start by tracking $\hat{\bar{x}}(t)$ a posited deterministic although initially unknown continuous path. We can then show that this tracking problem has a unique optimal control function

$$u^*(t) = -\frac{1}{r}B_\theta^T\big(\Gamma_j^{M\theta}(t)x_i + \beta_j^{M\theta}(t)\big) \quad \text{if } x^0 \in D_j^{M\theta}(\hat{\bar{x}}) \tag{18}$$

where $\Gamma_j^{M\theta}$, $\beta_j^{M\theta}$, $\delta_j^{M\theta}$ are the unique solutions of

$$\dot{\Gamma}_j^{M\theta} - \frac{1}{r}\Gamma_j^{M\theta}B_\theta B_\theta^T\Gamma_j^{M\theta} + \Gamma_j^{M\theta}A_\theta + A_\theta^T\Gamma_j^{M\theta} + qI_n = 0 \tag{19a}$$
$$\dot{\beta}_j^{M\theta} = \Big(\frac{1}{r}\Gamma_j^{M\theta}B_\theta B_\theta^T - A_\theta^T\Big)\beta_j^{M\theta} + q\hat{\bar{x}} \tag{19b}$$
$$\dot{\delta}_j^{M\theta} = \frac{1}{2r}(\beta_j^{M\theta})^T B_\theta B_\theta^T\beta_j^{M\theta} - \frac{1}{2}q\hat{\bar{x}}^T\hat{\bar{x}}, \tag{19c}$$

with the final conditions

$$\Gamma_j^{M\theta}(T) = M_j I_n,\ \ \beta_j^{M\theta}(T) = -M_j p_j,$$
$$\delta_j^{M\theta}(T) = \frac{1}{2}M_j p_j^T p_j,$$

$$D_j^{M\theta}(\hat{\bar{x}}) = \Big\{ x \in \mathbb{R}^n \text{ such that}$$
$$x^T \Gamma_{jk}^{M\theta} x + x^T \beta_{jk}^{M\theta}(\hat{\bar{x}}) + \delta_{jk}^{M\theta}(\hat{\bar{x}}) \leq 0, \, \forall k = 1, \dots, l \Big\} \tag{20}$$

and

$$\begin{aligned} \Gamma_{jk}^{M\theta} &= \Gamma_j^{M\theta}(0) - \Gamma_k^{M\theta}(0) \\ \beta_{jk}^{M\theta}(\hat{\bar{x}}) &= \beta_j^{M\theta}(0) - \beta_k^{M\theta}(0) \\ \delta_{jk}^{M\theta}(\hat{\bar{x}}) &= \delta_j^{M\theta}(0) - \delta_k^{M\theta}(0). \end{aligned} \tag{21}$$

The solution of Riccati equation (19a) is not the same for all the agents and depends on the initial preference vector M and the destination points. Hence, the basins of attraction are now regions delimited by quadratic form surfaces in $\mathbb{R}^n$. This fact complicates the structure of the operator that maps the tracked path to the mean as it is shown later. Thus, the proof of the existence of a fixed point requires an abstract, Banach space version of Brouwer's fixed point theorem, namely Schauder's fixed point theorem [22]. We define

$$\Psi_j^{M\theta}(\eta_1, \eta_2, \eta_3, \eta_4) = \Phi^T_{\Pi_j^{M\theta}}(\eta_1, \eta_2) B_\theta B_\theta^T \Phi_{\Pi_j^{M\theta}}(\eta_3, \eta_4)$$

where $\Pi_j^{M\theta}(t) = \frac{1}{r}\Gamma_j^{M\theta}(t) B_\theta B_\theta^T - A_\theta^T$ and $\Phi_{\Pi_j^{M\theta}}$ as in (9) where Π is replaced by $\Pi_j^{M\theta}$. The state of the generic agent is then

$$\begin{aligned} x(t) = \sum_{j=1}^{l} 1_{D_j^{M\theta}(\hat{\bar{x}})}(x^0) \Big\{ & \Phi^T_{\Pi_j^{M\theta}}(0,t) x^0 \\ & + \frac{M_j}{r} \int_0^t \Psi_j^{M\theta}(\sigma, t, \sigma, T) p_j \, \mathrm{d}\sigma \\ & - \frac{q}{r} \int_0^t \int_T^\sigma \Psi_j^{M\theta}(\sigma, t, \sigma, \tau) \hat{\bar{x}}(\tau) \, \mathrm{d}\tau \mathrm{d}\sigma \Big\}. \end{aligned}$$

Assumption 2: We assume that $\mathbb{E}\|x^0\| < \infty$.

The functions defined by (19a),(19b) and (19c) are continuous with respect to M and θ which are defined on compact sets. M, θ and x^0 are independent. Thus, under assumption 2, the mean of the population can be computed using Fubini-Tonelli's theorem as follows

$$\begin{aligned} \bar{x}(t) &= \mathbb{E}^{(M,\theta)}\Big(\mathbb{E}^{x^0} x(t) \Big) = \mathbb{E}^{(M,\theta)}\Big(\bar{x}^{M\theta}(t) \Big) \\ &= \sum_{j=1}^{l} \int_\Theta \int_W \int_{\mathbb{R}^n} 1_{D_j^{M\theta}(\hat{\bar{x}})}(x^0) \Big\{ \Phi^T_{\Pi_j^{M\theta}}(0,t) x^0 \\ &\quad + \frac{M_j}{r} \int_0^t \Psi_j^{M\theta}(\sigma, t, \sigma, T) p_j \, \mathrm{d}\sigma \\ & - \frac{q}{r} \int_0^t \int_T^\sigma \Psi_j^{M\theta}(\sigma, t, \sigma, \tau) \hat{\bar{x}}(\tau) \, \mathrm{d}\tau \mathrm{d}\sigma \Big\} \mathrm{d}P_0 \mathrm{d}P_M \mathrm{d}P_\theta. \end{aligned} \tag{22}$$

Equation (22) defines an operator G_p from the Banach space $(C([0,T], \mathbb{R}^n), \|.\|_\infty)$ into itself which maps the tracked path $\hat{\bar{x}}$ to the corresponding mean $\bar{x}$, considered as another potential tracked path. In the next theorem, we show that G_p has a fixed point.

Assumption 3 : We assume that P_0 is such that the P_0-measure of quadratic form surfaces is zero.

Theorem 5: G_p has a fixed point.

V. Nash Equilibrium

In the previous sections, we develop fixed point based decentralized strategies. To compute its strategy, each agent needs to know, in the uniform without initial preferences case, its own state and the initial distribution of the agents, and in the nonuniform with initial preferences case, its own state, the initial distribution of the population and the distribution of the dynamic parameters and vector of preferences. In the next theorem we show that these decentralized strategies are optimal in an approximate game theoretical sense with respect to the costs (2) and (17), in that they constitute an $\epsilon-$Nash equilibrium.

Theorem 6: If $\mathbb{E}\|x_i^0\|^2 < \infty$ then the decentralized strategies u_i^* for $i = 1, \dots, l$ constitute an ϵ - Nash equilibrium with respect to the costs $J_i\Big(x_i(u_i), \frac{1}{N}\sum_{j=1}^N x_j(u_j), x_i^0\Big)$, where $\epsilon = O(\epsilon_N)$ with ϵ_N converges to zero as N goes to infinity.

VI. Simulation Results

To illustrate the collective decision-making mechanisms (without initial preferences), we consider 500 agents initially drawn from a Gaussian distribution $N((-10 \;\, 0)^T, 5I_2)$ and moving in $\mathbb{R}^2$ according to the dynamics

$$A = \begin{bmatrix} 0 & 1 \\ 0.02 & -0.3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}.$$

Each agent makes a choice between the destination points $p_1 = (-39.3, -10)$, $p_2 = (-27, 9.5)$ or $p_3 = (0, 40)$ under a social effect of strength q (q is the coefficient that penalizes on the deviation from the mean). We start by the case where the social effect is negligible ($q = 0$). In this case, the unique fixed point of F_∞ is $\lambda = (0, 0.25, 0.75)$. Accordingly, quarter of the population go towards p_2 (see Fig.1 green balls) and the rest towards p_3 (see Fig.1 yellow balls). When the social effect increases to $q = 4$, some of the agents, that have chosen p_2 as a destination in the absence of a social effect, change their decision and follow the majority (see Fig.1 blue balls ($q = 4$)). In this case, we compute a fixed point of F_∞ using Broyden's method. We find $\lambda = (0, 0.16, 0.84)$. When the social effect reaches $q = 6$, a consensus to go towards p_3 occurs. All the agents, that went towards p_2 in the absence of a social effect, change their decision and go towards p_3 (see Fig.1 Blue balls). Moreover, the mean perfectly replicates the anticipated mean as shown in Figure 2 ($q = 4$).

VII. Conclusion

We consider in this paper a dynamic noncooperative game model for collective choice in multi-agent systems. In the simplest form of the game, the agents' final decisions are dictated by their initial positions. Subsequently, we generalize the model to include initial individual preferences towards the destination points. Moreover, we consider a nonuniform population. Using a mean field game theoretic framework, we establish that multiple $\epsilon-$Nash strategies may exist. To compute its individual control strategy, each agent needs to know its own state, the initial distribution of the agents,

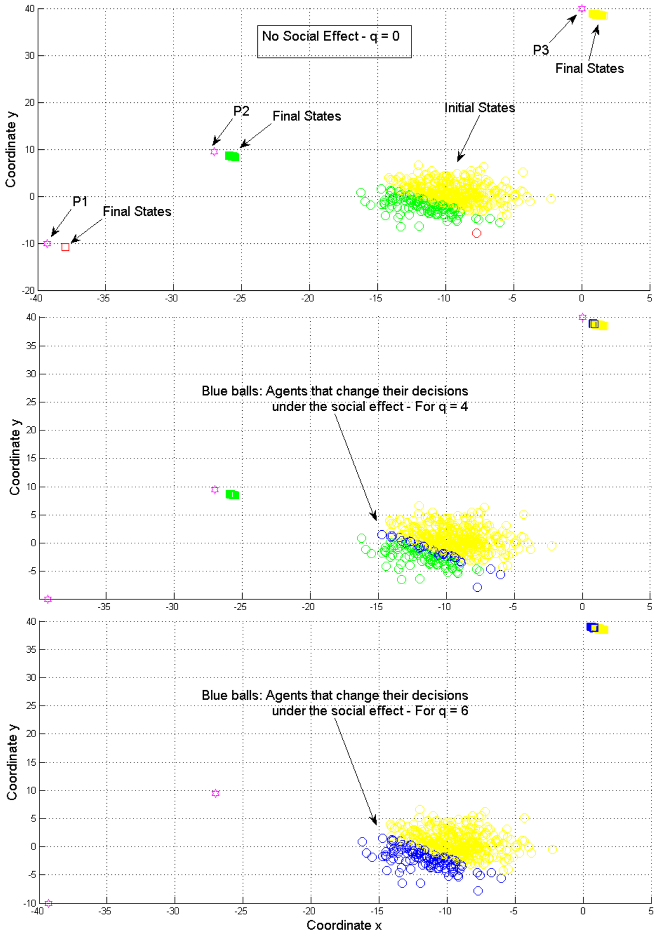

Fig. 1. Social Effect

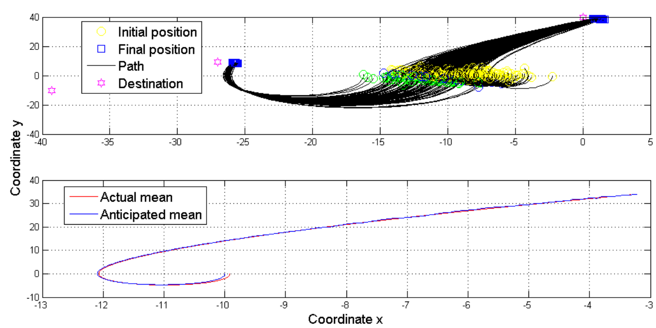

Fig. 2. $q = 4$

and in the nonuniform case with initial preferences, the distribution of the dynamic parameters and the distribution of the vector of preferences.

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