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A mean field route choice game model

Rabih Salhab, Jerome Le Ny and Roland P. Malhamé

Abstract— **We study a route choice game model, where a large number of drivers are circulating on a road network. Given an origin-destination pair, each driver tries to pick the shortest least congested route to minimize his/her travel time. We develop a mean field game based algorithm that generates the drivers' optimal choices and anticipates the evolution of their probability distribution on the network. The optimal choices, which constitute a Nash equilibrium in the limit of an infinite number of drivers, guide a generic driver to his/her destination with the most efficient road. Moreover, they define a maximum likelihood function that can be used to estimate the model's parameters. Our algorithm takes only the drivers' initial distribution as an input, which is typically provided to the drivers by navigation applications. Finally, we illustrate via a numerical scheme how the model can also be used to evaluate the performance of different network configurations. An example shows how adding a road link to an existing network might not improve the expected travel time of the drivers.**

I. INTRODUCTION

Route choice models [1] try to capture how drivers choose their routes between pairs of origins and destinations on a road network. They play a crucial role in transportation network planning. For example, they provide planners with predictions about the traffic load on the different road links of a network [2], allowing them to study whether adding a new link to an existing transportation network can improve the travel time of the drivers and reduce traffic congestion. The well-known Braess paradox [3] shows however that additional links can worsen the travel time of drivers when they can anticipate the level of congestion on the road network to play a non-cooperative game with each other, with each driver selfishly optimizing its own travel time. With the advance of information technology, it is similarly becoming clear that the effect of providing real-time information to the drivers must be better understood, as it can have negative consequences in a competitive environment [4]. A careful understanding of the interaction driver-information [5] is needed in order to decide the timing and nature of information to release to the drivers, for an uninformed driver is sometimes better for the overall performance of the network than an informed one [6], [7].

Drivers are rational decision makers and can anticipate the behavior of the other drivers when making an optimal decision. As a result, a model that seeks to explain how drivers move on a road network needs to conform with their rationality. For example, if the model assumes that the drivers are minimizing their travel times, then the latter should not be considered as exogenous variables independent of the drivers' future actions, but both a factor shaping their choices and a result of these choices at the same time. In other words, a rational driver needs to anticipate the amount of cars that will choose a certain link before picking it in order to minimize the additional travel time due to traffic congestion.

The main goal of this paper is to model and understand how a group of *rational* drivers on a road network make their route choices, given a set of origin-destination pairs. More precisely, we consider a route choice model involving a large number of drivers. Each driver anticipates the other drivers' future choices given the current information, to subsequently choose the shortest least congested route.

The main contributions of this paper are as follows:

1) We introduce a mean-field games approach to the route choice problem, which is a natural way of modeling the behavior of a large number of interacting rational decision makers.
2) Our model generates the transition probabilities of a generic driver on the network. These probabilities can be used for both estimating the model's parameters and guiding a driver to his/her destination with the most efficient road.
3) Given an initial distribution of drivers on a network, our model predicts the evolution of this distribution over time. This information is typically provided to drivers by navigation applications. In practice, one can measure the distribution of drivers on a periodic basis and use the model to predict its evolution between two consecutive measurements.
4) Our model can also be used to compare the performance of different candidate network configurations in terms of travel times, while taking into account the rationality of the individual drivers.

Route choice problems are studied in the literature from two different perspectives. The first is microscopic. It considers an individual driver and tries to understand how he/she makes his/her choice of route based on some personal attributes, such as the type of car he drives, and network related attributes, such as the travel times on the different road links and the network layout. These models are studied within the framework of discrete-choice theory in microeconomics [8]. They are logit-based models and differ in the assumptions made on correlations in travel times between different links. For example, Ben-Akiva [9] extended the multinomial-logit model to what he calls the nested logit model, in order

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to capture situations where the set of network links is partitioned into nests, such that intra nest link travel times are correlated, while links in different nests have independent travel times. Vovsha [10] generalized Ben-Akiva's model by proposing the Cross-nested logit model to allow for trans-nests correlations. Later, Gao *et al.* [5], [11] generalized these models to account for real-time information, where a driver revisits at each roads intersection his/her choice of the next road link based on some new information about the traffic conditions, such as the most recent travel times. These logit-based models consider the travel times on the different links as exogenous variables that do not depend on the future actions of the drivers. This corresponds to bounded rationality of the drivers, who according to these models act myopically, based on a travel time that does not reflect the current or future traffic conditions. The second perspective is macroscopic. It includes static and dynamic traffic assignment models [12]–[14]. These models are concerned with the macroscopic behavior of the drivers, i.e., the evolution of traffic flow on the road network.

Our model lies somewhere between the microscopic and macroscopic approaches. In fact, it starts by describing the individual choices of the drivers to anticipate the macroscopic behavior of the population, i.e., the evolution of the drivers' probability distribution on the network. This passage from the microscopic to the macroscopic levels is a result of the way the drivers interact with each others, and of the methodology we use to analyze our model, namely, *Mean Field Games* (MFG). Indeed, we consider a discrete time dynamic game with a finite number of states, which involves a large number of weakly coupled drivers/players, that is, an isolated individual strategy has a negligible impact of the others' strategies, while the mass behavior of the population has a considerable influence on the individual strategies. To analyze our game involving a large population of players, we follow the MFG methodology, which was introduced in a series of papers by Huang *et al.* [15]–[17], and independently by Lions and Lasry [18]–[20]. Discrete time finite state MFGs were studied later [21], [22]. To solve the game, the MFG methodology starts by considering the limiting case of a continuum of players, which can be described by two coupled forward-backward difference equations. The backward equation characterizes a generic player's best response to the macroscopic distribution of players, while the forward equation propagates the probability distribution of the players under these best response strategies. Candidate sustainable macroscopic behaviors/probability distributions, if they exist, are then computed by a fixed point argument. The best response strategies, when applied to the finite population, typically constitute approximate Nash equilibria.

The mathematical model of the route choice game is presented in Section II. In Section III, we solve the game via the MFG methodology, compute the drivers' optimal choices and show how to anticipate the drivers' probability distribution. Section IV discusses the main results of the paper. Section V reports some numerical simulation results, while Section VI presents our conclusion.

II. Mathematical Model

We model our problem as a finite-state dynamic game in discrete time. We consider a large number of drivers moving on a directed graph $\mathcal{G} = (E, N)$, where the edges $E = \{1, \dots, d\}$ represent the road links, and the nodes N the road intersections. We assume that the drivers are initially distributed on the road links according to a known probability distribution $\{\pi_0^i\}_{i\in E}$. A driver's goal is to move along the road network from its initial position at time $t = 0$ until reaching the destination edge $d \in E$ before the end of the time horizon $t = T$. He/She does so while minimizing his/her travel time. The state $x_t \in E$ of a driver at time $t \in \{1, \dots, T\}$ is its current position (edge). The density of the drivers at the edge i at time t is denoted by π_t^i. We assume that a driver at the edge x_t at time t can only move one edge per time period, i.e., $x_{t+1} \in \mathcal{N}(x_t)$, where $\mathcal{N}(i)$ is the set of outgoing edges from the sink node of edge i. A driver at the destination edge d at time t_0 stays at d for all $t \geq t_0$, that is, $\mathcal{N}(d) = \{d\}$. We explain these notions through road network example shown on Figure 1. This network consists of six road links $\{1, 2, 3, 4, 5, 6\}$, where the destination edge is 6. The sets of outgoing neighbors of edges $1, 2, 3, 5$ and 6 are respectively $\mathcal{N}(1) = \{3, 4\}$, $\mathcal{N}(2) = \{5\}$, $\mathcal{N}(3) = \{5\}$ and $\mathcal{N}(4) = \mathcal{N}(5) = \mathcal{N}(6) = \{6\}$.

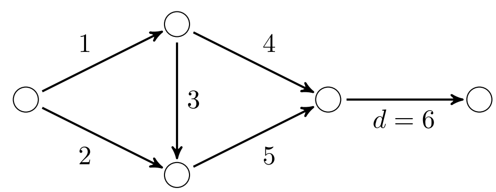

Fig. 1. A road network example.

The game is played as follows. We assume that a driver reaches his/her destination whenever he/she arrives at the origin node of d. A driver at time $t < T$ at the edge x_t, such that the destination edge $d \in \mathcal{N}(x_t)$, moves at time $t+1$ to d, i.e., d is an absorbing state. In this case, he/she does not pay a cost. Otherwise, a driver at edge i at time $t < T$ assigns a probability vector $\{P_t^{ij}\}_{j\in\mathcal{N}(i)}$ to the neighbor edges $\mathcal{N}(i)$. Subsequently, he/she moves with probability P_t^{ij} to the link j at time $t + 1$ and pays a per-step cost

$$c_t^{ij} = \alpha_{ij} P_t^{ij} \max(\pi_t^i, \epsilon) + h_j(\pi_{t+1}^j), \tag{1}$$

where $h_j : [0, 1] \mapsto \mathbb{R}_+$ is a continuous increasing function, and α_{ij} and ϵ are positive scalars. In other words, the probabilities $\{P_t^{ij}\}_{j\in\mathcal{N}(i)}$ are the control variables of a driver at edge i at time t. The term $\alpha_{ij} P_t^{ij} \max(\pi_t^i, \epsilon)$ is the "fork cost". It models the travel time that a driver spends at the fork while moving from link i to link j at time t. For $\epsilon = 0$, this cost increases with the fraction of drivers that make the same decision, i.e., $P_t^{ij}\pi_t^i$. Here, we assume in fact that the constant ϵ is small but strictly positive, for a technical reason that will become clear later (see Remark 2). The sensitivity coefficient α_{ij} depends on the physical layout of the fork. In the example of Figure 1, there are two $\alpha-$coefficients related to the fork that links edge 1 to edges 3 and 4, namely, α_{13} and α_{14}. If for example the number of lanes on link 1 dedicated to turning toward link 3 (this number is equal to 2 in Figure 2) is greater than that of the lanes dedicated to turning toward 4 (this number is equal to 1 in Figure 2), then the connection $1-3$ is capable of absorbing more cars than $1-4$, and $\alpha_{13} < \alpha_{14}$. In other words, if the group of cars that are on link 1 at time t splits evenly between links 3 and 4 at time $t+1$, then the travel time between 1 and 3 is less than that between 1 and 4.

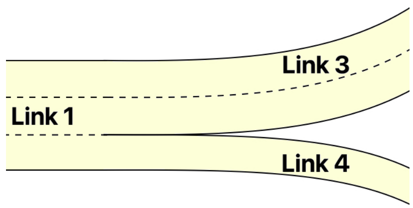

Fig. 2. Fork cost coefficients.

The term $h_j(\pi_{t+1}^j)$ is the "link cost", which models the travel time on link j. This term increases with the traffic congestion, i.e., the density of drivers π_{t+1}^j, and depends on the layout of link j. For example, we can have $h_j(\pi_{t+1}^j) = \delta_j / \max(\beta_j - \pi_{t+1}^j, \eta_j)$, where $\delta_j > 0$, $\beta_j \in [0,1]$, and η_j is a positive scalar much smaller than δ_j. In this case, δ_j/β_j models the travel time on link j without traffic congestion ($\pi_{t+1}^j = 0$), and δ_j/η_j the travel time on this link when the density of cars exceeds the capacity of link j, β_j. It should be noted here that δ_j/η_j is a large positive number. Finally, a driver at the end of the time horizon T pays a cost equal to $M > 0$ if he/she is not at the destination edge d. This final cost encourages the drivers to reach their destination before the end of the time horizon T.

In summary, our model aims at capturing the *individually* optimal route choice policies for the drivers on a road network. These drivers, considered rational, have to anticipate the strategies of the other drivers in order to avoid traffic congestion and arrive at the destination edge in minimum time. A solution in such situations is a Nash equilibrium, which is hard to compute for a large population. However, since we assume a homogeneous population of drivers that are only weakly coupled through their density via the cost (1), the methodology of MFGs, which we introduce next, offers a promising avenue for analyzing the game.

Remark 1: A potential generalization of this model is to assume a heterogeneous population of drivers with nonidentical origin-destination pairs and arrival time preferences driving on a time-varying graph. For clarity, we do not consider this case in the paper, although the corresponding analysis is similar to the one we present here.

III. Mean Field Equations

Following the mean field games methodology [21], we start by assuming a continuum of players whose probability distribution flow $\pi = \{\pi_t^i\}_{t,i}$ is deterministic and assumed known for now. Later in this section, we explain how to compute this flow by using a fixed point argument capturing the fact that the drivers must collectively reproduce this flow when they individually react optimally to it. With the distribution flow π known to all players, a generic driver faces an optimal control problem and chooses its transition probabilities P_t^{ij} to minimize its cost by solving the following dynamic program [23],

$$\begin{aligned} V_t^d &= V_{t+1}^d, && t < T, \\ V_t^i &= \min_{P_t^i \in S_i} \sum_{j \in \mathcal{N}(i)} \left(c_t^{ij} + V_{t+1}^j \right) P_t^{ij}, && i \neq d, t < T, \end{aligned} \quad (2)$$

with $V_T^d = 0$ and $V_T^i = M$, for $i \neq d$. Here, the set S_i is the set of probability measures $P_t^i = (P_t^{i1}, \ldots, P_t^{id})$ with support $\mathcal{N}(i)$, and V_t^i is the optimal cost-to-go at time t given the current state $x_t = i$. Note that the cost V_t^i depends on the probability distribution flow π through the per-step cost c_t^{ij} defined in (1).

Given the probability distribution flow π, and assuming the cost-to-go V known, a generic driver's best response $\{\bar{P}_t^{ij}\}_{t,i,j}$ to π solves the following convex program,

$$\begin{aligned} &\min_{P_t^i \in \mathbb{R}^d} \sum_{j=1}^{d} \left(\alpha_{ij} P_t^{ij} \max(\pi_t^i, \epsilon) + h_j(\pi_{t+1}^j) + V_{t+1}^j \right) P_t^{ij} \\ &\text{s.t. } P_t^{ij} \geq 0, \sum_{j \in \mathcal{N}(i)} P_t^{ij} = 1, \text{ and } \sum_{j \notin \mathcal{N}(i)} P_t^{ij} = 0. \end{aligned} \quad (3)$$

The convex program (3) has a unique "water filling"-type solution [24], which we give in the following theorem.

Theorem 1: The convex program (3) has a unique solution:

$$\begin{aligned} \bar{P}_t^{ij} &= \frac{\max\left(v_t^{ij} - V_{t+1}^j, 0 \right)}{2\alpha_{ij} \max(\pi_t^i, \epsilon)}, && \text{for } j \in \mathcal{N}(i), \\ \bar{P}_t^{ij} &= 0, && \text{for } j \notin \mathcal{N}(i), \end{aligned} \quad (4)$$

where

$$v_t^{ij} = -\lambda_t^i - h_j(\pi_{t+1}^j) \quad (5)$$

and λ_t^i is the unique solution of

$$g_t^i(\lambda_t^i) \triangleq \sum_{j \in \mathcal{N}(i)} \frac{\max\left(-\lambda_t^i - h_j(\pi_{t+1}^j) - V_{t+1}^j, 0 \right)}{2\alpha_{ij} \max(\pi_t^i, \epsilon)} = 1. \quad (6)$$

Proof: The drivers are constrained to move on the graph $\mathcal{G}$. As a result $\bar{P}_t^{ij} = 0$ for $j \notin \mathcal{N}(i)$. Moreover, the unique solution $\bar{P}_t^i$ of the convex program (4) satisfies the following KKT conditions [25, Section 5.5.3]:

$$\mu_t^j + \lambda_t^i = -2\alpha_{ij} \bar{P}_t^{ij} \max(\pi_t^i, \epsilon) - h_j(\pi_{t+1}^j) - V_{t+1}^j \quad (7)$$

$$\bar{P}_t^{ij} \geq 0, \quad (8)$$

$$\sum_{j \in \mathcal{N}(i)} \bar{P}_t^{ij} = 1 \quad (9)$$

$$\mu_t^j \geq 0, \quad (10)$$

$$\mu_t^j \bar{P}_t^{ij} = 0, \quad (11)$$

$\forall j \in \mathcal{N}(i)$, for some $\lambda_t^i \in \mathbb{R}$ and $\mu_t^j \in \mathbb{R}$. By multiplying both sides of equation (7) by μ_t^j and noting (11), one can deduce (4). The equality $g_t^i(\lambda_t^i) = 1$ follows from (4) and (9). It remains to prove that there exists a unique solution to $g_t^i(\lambda_t^i) = 1$. In fact, g_t^i is a continuous piecewise affine strictly decreasing function from $\mathbb{R}$ onto $[0, \infty)$, hence (6) has a unique solution λ_t^i. ■

According to the policies (4), there exists at each step time t a set of threshold costs $\{v_t^{ij}\}_{j\in\mathcal{N}(i)}$, such that a driver at time t at edge i will not move at time $t+1$ to a "costly" edge j with expected cost-to-go greater than the threshold cost v_t^{ij}. It should be noted that in Theorem 1 we do not consider the trivial case where $d \in \mathcal{N}(i)$. Indeed, in this case $\bar{P}_t^{id} = 1$ and $\bar{P}_t^{ij} = 0$, for $j \neq d$.

Having computed the drivers' best responses to π, we turn to the problem of computing a sustainable probability distribution flow π and the corresponding cost-to-go V. A sustainable distribution flow is such that it is collectively replicated by the drivers when they optimally respond to it. Hence, π and V must satisfy the following coupled forward-backward difference equations, which constitute the mean field equations,

$$V_t = \mathcal{F}_1(V_{t+1}, \pi), \qquad V_T = (M, \dots, M, 0), \tag{12}$$

$$\pi_{t+1} = \mathcal{F}_2(\pi_t, V), \qquad \pi_0 = (\pi_0^1, \dots, \pi_0^d), \tag{13}$$

where $\pi = \{\pi_t^i\}_{t,i}$, $V = \{V_t^i\}_{t,i}$, $V_t = (V_t^1, \dots, V_t^d)$ and $\pi_t = (\pi_t^1, \dots, \pi_t^d)$. Equation (12) is the dynamic program equation (2) where we replace $\{P_t^{ij}\}_{t,i,j}$ by the best responses $\{\bar{P}_t^{ij}\}_{t,i,j}$ to the flow π. Equation (13) can be written more explicitly as

$$\pi_{t+1}^j = \sum_{i=1}^{d} \bar{P}_t^{ij} \pi_t^i, \ \forall j, \tag{14}$$

and describes the evolution of the drivers' distribution under their best responses $\{\bar{P}_t^{ij}\}_{t,i,j}$. Note here that $\bar{P}_t^{ij}$ depends on π and V.

A flow π is sustainable if and only if it is a fixed point of the map $f = f_2 \circ f_1$, described as follows. The function f_1 is defined from the set of probability distributions flows S^{T+1}, with $S = \{(y_1, \dots, y_d) \in \mathbb{R}^d | \sum_{i=1}^d y_i = 1, y_i \geq 0, \forall 1 \leq i \leq d\}$, to the set $\mathbb{R}^{(T+1)\times d}$, such that $f_1(\pi)$ is the cost-to-go function V when a generic driver optimally respond to π, i.e., $f_1(\pi)$ is equal to the unique solution V of the backward difference equation (12). The function f_2 is defined from $\mathbb{R}^{(T+1)\times d}$ to S^{T+1}, such that $f_2(V)$ is the unique solution π of (13). The following theorem shows that there always exists a pair (π, V) that satisfies the mean field equations (12)-(13), or equivalently, a fixed point π of f.

Theorem 2: There exists a sustainable probability distribution flow and corresponding cost-to-go, i.e., a pair (π, V) satisfying (12)-(13).

Proof: The set S^{T+1} is a nonempty convex compact subset of $\mathbb{R}^{(T+1)d}$. Hence, it is sufficient to show that $f_2 \circ f_1$ is continuous, in which case Brouwer's fixed point theorem [26, Section V.9] guarantees the existence of a fixed point flow π of $f_2 \circ f_1$. In the following, we show that λ_t^i, the unique solution of (6), is continuous with respect to V_{t+1} and π, which implies that the best response (4) is continuous with respect to V_{t+1} and π. This, combined with (2) and (14), implies that $f_2 \circ f_1$ is continuous. The function g_t^i defined in (6) is continuous piecewise linear, with break points $(b_j, g_t^i(b_i))$, for $j \in \mathcal{N}(i)$, where $b_j = -h_j(\pi_{t+1}^j) - V_{t+1}^j$. Therefore, it is sufficient to show that the break points change continuously with V_{t+1} and π. Let us consider g_t^i defined in (6) as a function of $(\lambda_t^i, V_{t+1}, \pi)$. Then, $g_t^i(\lambda_t^i, V_{t+1}, \pi)$ is a continuous function of $(\lambda_t^i, V_{t+1}, \pi)$. Moreover, b_j is a continuous function of (V_{t+1}, π). Hence, the break points $(b_j, g_t^i(b_i))$, which are equal to $(b_j(V_{t+1}, \pi), g_t^i(b_j(V_{t+1}, \pi), V_{t+1}, \pi))$ are continuous functions of (V_{t+1}, π). This proves the result. ■

Remark 2: If $\epsilon = \pi_t^i = 0$, then one of the $\bar{P}_t^{ij}$ is equal to one and the others are zero, and the function f defined above Theorem 2 is no longer continuous. Hence, we assume $\epsilon > 0$ to guarantee the existence of a fixed point π.

IV. Discussion

A. Computation of a Nash equilibrium

Theorem 2 guarantees the existence of at least one infinite population Nash equilibrium $\{\bar{P}_t^{ij}\}_{t,i,j}$, which can be computed by only knowing the initial probability distribution of the drivers π_0, as follows. The first step is to find a fixed point probability distribution flow π of the map f, defined above Theorem 2. Here, we can apply Broyden's method [27] to the function $f(\pi) - \pi$. The key feature of this quasi-Newton method is that it avoids the computation of the Jacobian matrix and its inverse, which is advantageous when working in high dimensional spaces, such as the set of distribution flows $S^{T+1} \subset \mathbb{R}^{(T+1)d}$ in our case. Once a fixed point π is computed, the second step consists in propagating backwards the dynamic program equation (12), and then computing the best responses (4). It should be noted here that the only information needed to find an equilibrium is π_0, which is used in equation (13) to compute the value of f at any distribution flow π.

B. Parameters estimation and practical use of the model

We discuss briefly in this section the problem of estimating the model parameters and how our model can be used in practical situations. These two problems will be studied in greater detail in future work. We denote by θ the vector of parameters to be estimated, for example $\theta = \{\alpha_{ij}, \beta_j, \delta_j, \eta_j\}_{i,j}$ when $h_j(\pi_{t+1}^j) = \delta_j / \max(\beta_j - \pi_{t+1}^j, \eta_j)$. The dataset D consists of the observed paths of n drivers, i.e., $D = \{i_0^j i_2^j \dots i_T^j, 1 \leq j \leq n\}$, where the $j-$th driver's path $i_0^j i_2^j \dots i_T^j$ is the sequence of road links taken by the driver between $t = 0$ and $t = T$. In order to have uncorrelated data, we assume that the n paths are observed during n non-overlapping time intervals. Given the dataset D, the maximum likelihood estimation method [28] consists in finding the parameters θ that maximize the probability of occurrence of D. According to our model, the probability that the $j-$th driver takes the path $i_0^j \dots i_T^j$ is $F_j(\theta) = \bar{P}_0^{i_0^j i_1^j} \times \dots \times \bar{P}_{T-1}^{i_{T-1}^j i_T^j}$, where F_j depends on θ through the best responses (4). As a result, the probability of occurrence of the observed dataset D is equal to $F_D(\theta) = \prod_{j=1}^{n} F_j(\theta)$. As shown in IV-A, to compute the best responses for a given θ (and as result $F_D(\theta)$), one needs to find a fixed point distribution flow π. A potential algorithm to find a maximum of F_D would include two loops; (i) an internal loop that computes the value of F_D as discussed in IV-A, and (ii) an external loop that finds the maximizer of F_D, for example, the gradient descent algorithm [25].

Once the parameters are estimated, the model can be used to predict how an initial distribution of drivers evolves on the road network. This reduces the amount of collected data needed to forecast the traffic conditions on the network. In fact, instead of continuously measuring the distribution of the cars on the different links, one can measure their density on a periodic basis, and use our model to predict its evolution between two sets of measurements. The periodic measurements are needed here in order to compensate for the prediction error due to the fact that the MFG methodology assumes an infinite number of perfectly rational drivers. The model can also be useful in transportation planning, for example, to compare the performance of different road configurations in terms of travel times, which are somehow reflected in the cost-to-go given by (12). On the individual level, the best responses (4) can be used as "navigators" to guide a driver to his/her destination using the most efficient road. These navigators are local, i.e., located in the drivers' cars, and need only to measure the current position of the car, and be fed periodically with the actual distribution of the cars on the network. As a result, they are advantageous from both an implementation efficiency and from a privacy point of view.

V. Simulation Results

In this section, we illustrate through a numerical example how our model can be used to predict the evolution of the probability distribution of the drivers on a network, to compare the performance of two network configurations, and to guide optimally an individual driver to his/her destination.

We consider a network consisting of 13 road links. The network layout is shown in Figure 3. We assume that the link cost is given by $h_j(\pi^j_{t+1}) = \delta_j / \max(\beta_j - \pi^j_{t+1}, \eta_j)$, and that at time $t = 0$, 30% of the cars are at edge 1, 30% at edge 3 and the rest at edge 2. The drivers need to move from their initial positions to reach the destination edge $d = 13$ before $T = 7$. We assume that $\delta_5/\beta_5 = \delta_7/\beta_7 = 10$, and $\delta_j/\beta_j = 3$ for the other edges. This means that, without traffic congestion, the travel time on links 5 and 7 is $10/3$ longer than that on the other links. $\eta_j = 0.01$ and the β coefficients are $\beta_{10} = 0.9$, $\beta_{11} = \beta_4 = 0.1$, and $\beta_j = 0.3$ for the other edges. This means that the capacity of link 10 is 9 times that of links 4 and 11, and 3 times the capacity of the other links. The α_{ij} coefficients are equal to 2 and the final cost M at an edge $i \neq d$ is equal to 5.

We consider two scenarios. In the first one, we assume that link 10 does not exist, while in the second scenario link 10 exists. Following the numerical method discussed in Section IV-A, we compute the probability distribution flow π and the optimal cost-to-go functions V for both scenarios. Figures 4 and 5 show respectively the evolution of the cars' distribution without and with link 10. As shown in Figure 5, 18% of the cars use the link 10 at time $t = 3$. This additional link reduces the expected cost-to-go of a driver initially at edge 1 from 61 to 60. For drivers initially at edges 2 and 3, this cost remains approximately the same with and without link 10. This means that the additional link 10 does not improve the total travel time of the drivers. Finally, Figure 6 shows the sample path (red dashed line) of a driver on the network with and without link 10.

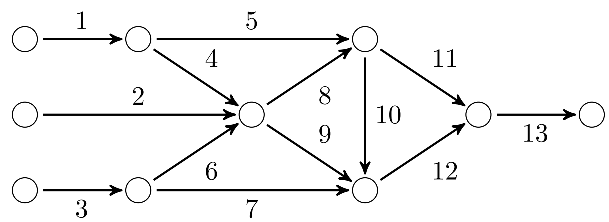

Fig. 3. Network layout

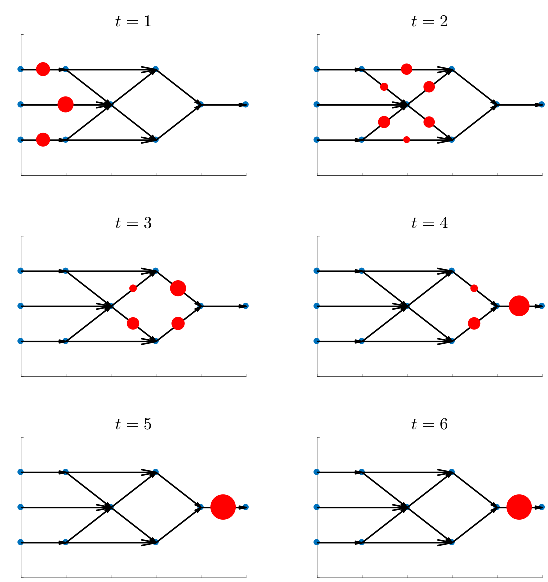

Fig. 4. Evolution of the cars' probability distribution without link 10. The radius of the red balls at edge j is proportional to the density of cars on that edge.

VI. Conclusion

We consider in this paper a route choice game model, where a large number of rational selfish drivers are choosing their routes to reach their destination in minimum time. We develop via the MFG methodology a set of Nash strategies, and suggest a numerical method to anticipate the evolution of the drivers' distribution on the network. One needs only

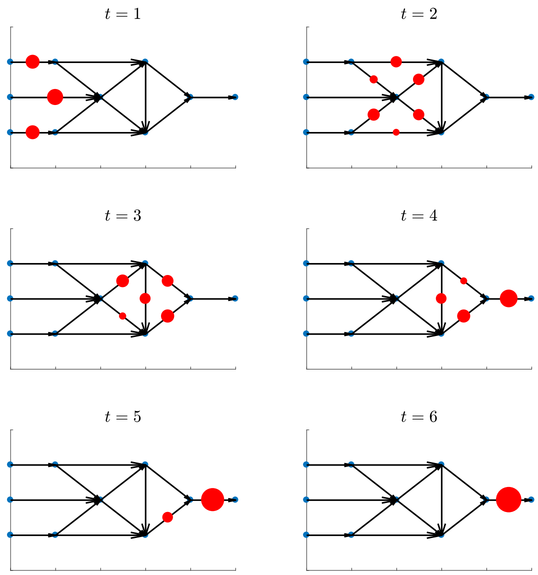

Fig. 5. Evolution of the cars' probability distribution with link 10. The radius of the red balls at edge j is proportional to the density of cars on that edge.

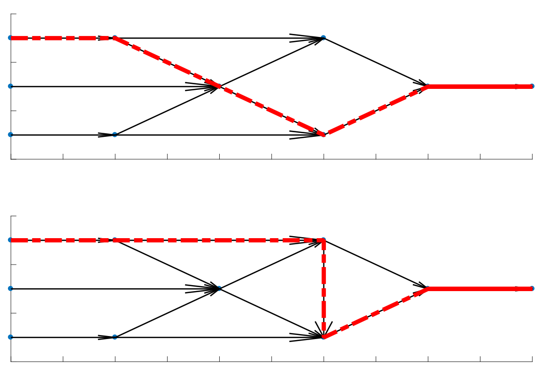

Fig. 6. Sample path of a driver on the road network with and without link 10.

to know the initial distribution of the drivers to run this numerical algorithm. This limits the amount of collected information needed to forecast the traffic conditions on the network. Finally, the Nash strategies generated by our model describe the transition probabilities of the drivers. They are of great importance when considering the estimation of the model's parameters, which will be the subject of our future work.

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