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A Dynamic Collective Choice Model With An Advertiser

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Abstract— **We consider within the framework of Mean Field Games theory a dynamic discrete choice model with an advertiser, where a large number of minor agents (e.g., consumers) are choosing between two predefined alternatives while influenced by social and advertisement effects. For example, in schools, teenagers' decisions to smoke are considerably affected by their peers (social effect), as well as the ministry of health campaigns against smoking (advertisement effect). The advertiser is "Stackelbergian", in the sense that it makes its decision first and then the consumers make their choices. We show for a continuum of minor agents that there exists a Stackelberg solution. Moreover, when the minor agents are initially uniformly distributed on a line segment, we give an explicit form for the solution and characterize it by a scalar describing the way the population of agents splits between the destination points.**

I. Introduction

Discrete choice models were initially developed in microeconomics to analyze human choice behavior in the face of a finite set of alternatives, e.g., the choice of a mode of transportation [1], of a residential location [2], smoking decision in schools [3], etc. These choices are mainly influenced by the others' choices (social or peers effect), as well as by some personal factors, such as the financial situation in the residential location example. In some situations, a third factor considerably affects the individual choices, namely the "advertisement effect". We call advertisement the effort exerted by an advertiser to induce individuals to choose one alternative over the others. For example, in its intent to reduce the percentage of smokers among teenagers, the government makes some investments in the form of campaigns against smoking to encourage the teenagers not to smoke. On the other hand, a teenager's decision to smoke is considerably affected by her peers' decisions. A related topic in differential game theory are the advertising competition models [4]. In these models, the consumers are not part of the game.

In this paper, we model within the framework of Mean Field Games (MFG) theory situations where a large number of agents (e.g., consumers) are making a choice among two alternatives, while taking into account the social and advertisement effects. The latter is produced by a dominating agent/advertiser advertising for one of the choices. The advertiser is "Stackelbergian" [5], that is it makes its decision first, with the consumers deciding afterwards, i.e. advertisement precedes consumption. We seek a Stackelberg solution. In case of initially uniformly distributed consumers, we describe by a scalar the way the population splits between the alternatives. This scalar is a fixed point of a well defined finite dimensional operator.

The MFG methodology, which we follow in this paper, is concerned with a class of dynamic games involving a large number of agents interacting through the mass effect of the group. It posits at the outset an infinite population to which one can ascribe a deterministic although initially unknown macroscopic behavior (i.e. a given flow of population probability distributions also known as the mean field). In view of the vanishing influence of isolated individuals, a generic agent's best response is then described by a Hamilton-Jacobi-Bellman (HJB) equation propagating backwards and parameterized by the macroscopic behavior. In turn, the macroscopic behavior satisfies a forward Fokker-Planck (FP) equation parameterized by the generic agents' best response. Candidate sustainable macroscopic behaviors are then computed as the fixed points, if they exist, of a suitable macroscopic to macroscopic behavior operator. The corresponding best responses when applied to the practical (finite population) situation, constitute, under adequate conditions, an approximate Nash equilibrium ($\epsilon-$Nash equilibrium) [6], [7].

Definition 1: Consider N agents, a set of strategy profiles $S = S_1 \times \cdots \times S_N$ and, for each agent k, a payoff function $J_k(u_1, \ldots, u_N)$, $\forall (u_1, \ldots, u_N) \in S$. A strategy profile $(u_1^*, \ldots, u_N^*) \in S$ is called an $\epsilon-$Nash equilibrium with respect to the costs J_k, if there exists an $\epsilon > 0$ such that for any fixed $1 \leq i \leq N$, for all $u_i \in S_i$, we have $J_i(u_i, u_{-i}^*) \geq J_i(u_i^*, u_{-i}^*) - \epsilon$, where $u_{-i}^* = (u_1^*, \ldots, u_{i-1}^*, u_{i+1}^*, \ldots, u_N^*)$.

The MFG theory was originally developed in a series of papers by Huang *et al.* [6]–[8], and independently by Lions and Lasry [9]–[11]. Recently, Bensoussan *et al.* developed in [12], [13] a Stackelberg MFG model where a dominating agent plays first and then a large number of minor agents make their decisions sequentially. In this case, the agents seek a Stackelberg solution [5], [14]. This is in contrast to the Nash equilibria sought in the minor-major agent games as originally introduced by Huang [15].

Definition 2: Consider $N + 1$ agents, a set of strategy profiles $S = S_0 \times \cdots \times S_N$, and for each agent k, a payoff function $J_k(u_0, \ldots, u_N)$, $\forall (u_0, \ldots, u_N) \in S$. Suppose that agent 0 is the dominating agent. A strategy profile $(u_0^*, \ldots, u_N^*) \in S$ is called a Stackelberg solution w.r.t. the costs J_k, if there exists a map T from S_0 to $S_1 \times \cdots \times S_N$, such that for all $u_0 \in S_0$, $T(u_0)$ is a Nash Equilibrium w.r.t. J_k, $k = 1, \ldots, N$, and $u_0^* = \min_{u_0 \in S_0} J_0(u_0, T(u_0))$, with $T(u_0^*) = (u_1^*, \ldots, u_N^*)$.

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In the model considered in this paper, the only randomness lies in the agents' initial conditions, and the control strategies, while expressed as state feedback laws, correspond in effect to open loop policies. Unlike [12], [13], in this paper, the dominating agent's optimal control problem involves a dynamic constraint that depends non-linearly on the minor agents' macroscopic behavior and the dominating agent's state. An explicit Stackelberg solution is thus possible only in some special cases, for example, uniform initial distribution on a line segment of the minor agents.

The main contribution of this paper is to introduce an "advertising" model in which both the consumers and advertiser are part of the game. Moreover, it describes the way the population of consumers splits between the alternatives under the social and advertisement effects. The mathematical model is introduced in Section II. In Section III, we solve the limiting problem and show the existence of a Stackelberg solution. In Section IV, we consider the case of initially uniformly distributed minor players and give an explicit form of the solution. In this case, we describe by a scalar λ the way the population splits between the alternatives under the social and advertisement effects. In Section V, we provide some numerical simulation results, while Section VI presents our conclusions.

Remark 1: For the proofs of theorems and lemmas, we refer the reader to [16].

A. Notation

The following notation is used throughout the paper. We fix a generic probability space $(\Omega, \mathcal{F}, P)$ and denote by $\mathbb{E}(X)$ the expectation of a random variable X. The indicator function of a subset X is denoted by 1_X. The transpose of a matrix M is denoted by M'. We denote by I_k the identity $k \times k$ matrix. We denote the matrix product MM' by M^2. Throughout this paper, $L_2([0,T], \mathbb{R}^m)$ is endowed with the inner product $\langle f, g\rangle = \int_0^T f(t)' g(t)\mathrm{dt}$, and the induced norm is denoted by $\|.\|_2$.

II. Mathematical Model

We consider a dynamic non-cooperative game involving N minor agents and one dominating agent/advertiser with respective dynamics

$$\dot{x}_i = Ax_i + Bu_i \qquad i = 1, \dots, N, \qquad (1a)$$

$$\dot{y} = A_0 y + B_0 v, \qquad (1b)$$

where $x_i \in \mathbb{R}^n$, x_i^0 and $u_i \in L_2([0,T], \mathbb{R}^m)$ are the state, initial state and control input of the minor agent i, while $y \in \mathbb{R}^{n_1}$, y^0 and $v \in L_2([0,T], \mathbb{R}^{m_1})$ are the state, initial state and control input of the dominating agent. We assume that the initial conditions x_i^0, $i = 1, \dots, N$, are independent and identically distributed (i.i.d.) random vectors on some probability space $(\Omega, \mathcal{F}, P)$ with distribution P_0.

The minor and dominating agents are associated with the following individual (non-convex) cost functions:

$$J_i(u,v) = \mathbb{E}\Big[\int_0^T \Big\{\frac{q}{2}\big\|x_i - \alpha\bar{x} - K(p_2)y\big\|^2 + \frac{r}{2}\big\|u_i\big\|^2\Big\}\mathrm{dt} + \frac{M}{2}\min_{j=1,2}\Big(\big\|x_i(T) - p_j\big\|^2\Big)\Big], \qquad (2a)$$

$$J_0(v,u) = \mathbb{E}\Big[\int_0^T \frac{r_0}{2}\big\|v\big\|^2\mathrm{dt} + \frac{M_0}{2}\big\|\bar{x}(T) - p_2\big\|^2\Big], \qquad (2b)$$

for $i = 1, \dots, N$, where $u = (u_i, u_{-i})$, $\bar{x} = 1/N\sum_{i=1}^N x_i$, $\alpha \geq 0$, $q, r, r_0, M, M_0 > 0$, and $p_j \in \mathbb{R}^n$, $j = 1, 2$, are the possible ultimate destination choices. The matrix $K(p_2) \in \mathbb{R}^{n\times n_1}$ is a function of the destination point p_2. M is a large number, which incites the final state to be close to one of the two destinations at time T.

Example 1: Consider a group of N teenagers choosing before a time T between smoking (p_1) or not smoking (p_2). At time t, teenager i's smoking inclination is modeled by a variable $x_i(t) \in [-1, 1]$, where the value -1 correspond to a nonsmoker, while 1 represents a full smoker. The effort exerted by i at time t to change its status is modeled by $u_i(t) \in \mathbb{R}$. For example, $|u_i|$ would represent the amount of money paid by i to increase (extra cigarettes) or decrease (medical treatment) its status. On the other hand, the government investments against smoking is modeled by a variable $v \in \mathbb{R}$. The variable y represents the effectiveness of the advertising investment. The influence exerted by the advertisement on the teenagers' smoking status is modeled by $K(p_2)y$, where $K(p_2) := p_2 =$ "Do Not Smoke" $= -1$. A teenager, in the process of choosing between not smoking or smoking, minimizes the cost (2a), which penalizes along the path the deviation from the peers smoking status $\bar{x}$ and the government nonsmoking advertisement $K(p_2)y$, as well as the effort to change the smoking status. Moreover, the teenager should be by time T a smoker (p_1) or nonsmoker (p_2)) lest he/she be considered undecisive by its peers. Thus, lack of a decision by time T is strongly penalized in the final cost. On the other hand, the government tries to minimize its advertisement investments (the running cost of (2b)), and should convince by time T the teenagers to be nonsmokers. Failure to sway a majority of teenagers away from smoking results in a strong penalty in the final cost.

III. Mean Field Stackelberg Competition

In a Stackelberg competition, the dominating agent (DA) plays first and then the minor agents (mA) make their decisions. The agents solve the game as follows. On the one hand, given the DA strategy v, the mA play a Nash equilibrium with respect to their individual costs $J_i(u_i, u_{-i}, v)$, $i = 1, \dots, N$. On the other hand, to compute its optimal strategy, the DA constructs a function that maps its strategies v to the corresponding mA Nash equilibrium $(u_i^*(v), u_{-i}^*(v))$, if it exists uniquely. Subsequently, by implementing this map in its cost, the DA computes its optimal strategy by minimizing $J_0(v, u_i^*(v), u_{-i}^*(v))$.

In view of (2a)-(2b), the DA/mA interact with the mA through the mean field term $\bar{x}$. An efficient methodology to solve dynamic games involving a large number of weakly coupled agents is the MFG approach. We start by assuming a continuum of mA to which one can ascribe a deterministic although initially unknown mean trajectory $\bar{x}$. Then, the limiting game consists of (i) a representative (generic) agent of state x and initial state x_0, where x_0 is a random vector of distribution P_0, and (ii) the DA defined in (1b). The state x satisfies (1a). The generic agent and DA limiting cost functionals are respectively,

$$J(u,\bar{x},v) = \int_0^T \left\{\frac{q}{2}\left\|x - \alpha\bar{x} - K(p_2)y\right\|^2 + \frac{r}{2}\left\|u\right\|^2\right\}\mathrm{d}t + \frac{M}{2}\min_{j=1,2}\left(\left\|x(T) - p_j\right\|^2\right), \tag{3a}$$

$$\bar{J}_0(v,\bar{x}) = \int_0^T \frac{r_0}{2}\left\|v\right\|^2\mathrm{d}t + \frac{M_0}{2}\left\|\bar{x}(T) - p_2\right\|^2, \tag{3b}$$

where $\bar{x} = \mathbb{E}[x]$, in view of the assumed independence of the agents' random initial conditions. The cost functionals (3a)-(3b) are those of the mA/DA, where the average of the minor agents is replaced by an assumed given deterministic trajectory $\bar{x}$.

In the following subsection, we show that there exists, for any DA strategy v, a mA Nash equilibrium. Moreover, we characterize each equilibrium by a scalar λ describing the way population of mA splits between the alternatives under the social effect and the advertisement effort v.

A. mA Nash Equilibrium

Given the DA strategy v associated with an influence state trajectory $y(t)$, $t \in [0,T]$, we start by computing the generic agent's best response to $\bar{x}$. The cost function (3a) can be written as the minimum of two independent Linear Quadratic Regulator (LQR) optimal tracking problems, each associated with one of the two destination points. $J(u,\bar{x},v) = \min\left(J^1(u,\bar{x},v), J^2(u,\bar{x},v)\right)$, where

$$J^j(u,\bar{x},v) = \int_0^T \left\{\frac{q}{2}\left\|x - \alpha\bar{x} - K(p_2)y\right\|^2 + \frac{r}{2}\left\|u\right\|^2\right\}\mathrm{d}t + \frac{M}{2}\left\|x(T) - p_j\right\|^2,$$

with $j = 1,2$. Accordingly, we define the basin of attraction $D(\bar{x},y)$, such that if the generic agent is initially in $D(\bar{x},y)$, then the LQR optimal tracking problem corresponding to p_1 is the less costly, and the generic agent goes towards p_1. Otherwise, it goes towards p_2. The representative agent's best response u^* satisfies [17]

$$\begin{aligned} u^* &= -\frac{1}{r}B'n \\ -\dot{n} &= A'n + q\left(x - \alpha\bar{x} - K(p_2)y\right), \end{aligned} \tag{4}$$

with $n(T) = M\left(x(T) - p_1 1_{D(\bar{x},y)}(x_0) - p_2 1_{D(\bar{x},y)^c}(x_0)\right)$ and

$$\begin{aligned} D(\bar{x},y) &= \left\{x_0 \in \mathbb{R}^n | J^{1,*}(x_0,\bar{x},v) - J^{2,*}(x_0,\bar{x},v) \le 0\right\} \\ &= \left\{x_0 \in \mathbb{R}^n | \beta' x_0 \le \delta + \Delta\left(\alpha\bar{x} + K(p_2)y\right)\right\}, \end{aligned}$$

where $J^{j,*}(x_0,\bar{x},v)$ is the optimal cost of the LQR optimal tracking problem associated with p_j, Δ is a linear form on $L_2([0,T],\mathbb{R}^n)$, such that for all $x \in L_2([0,T],\mathbb{R}^n)$

$$\Delta(x) = \frac{Mq}{r}(p_1 - p_2)' \int_T^0 \int_T^\eta \phi(\eta,T)'B^2\phi(\eta,\sigma)x(\sigma)\,\mathrm{d}\sigma\mathrm{d}\eta, \tag{5}$$

ϕ is the state-transition matrix (STM) of $\frac{1}{r}\Gamma(t)B^2 - A'$ and

$$\begin{aligned} \dot{\Gamma} &= \frac{1}{r}\Gamma B^2\Gamma - \Gamma A - A'\Gamma - qI_n,\ \Gamma(T) = MI_n \\ \beta &= M\phi(0,T)(p_2 - p_1) \\ \delta &= \frac{1}{2}M(\|p_2\|^2 - \|p_1\|^2) + \frac{M^2}{2r}p_2' \int_T^0 \left(\phi(\eta,T)'B\right)^2\mathrm{d}\eta\, p_2 \\ &\quad - \frac{M^2}{2r}p_1' \int_T^0 \left(\phi(\eta,T)'B\right)^2\mathrm{d}\eta\, p_1. \end{aligned}$$

Given the macroscopic behavior $\bar{x}$ and the DA influence function $y(t)$, the generic agent's best response is uniquely determined. Now, for a given $y(t)$ trajectory, we study the existence of a consistent macroscopic behavior $\bar{x}$, i.e., such that $\bar{x}$ is the mean of the generic agent's state at the ($\bar{x}$, $y(t)$ dependent) equilibrium. By taking the expectations of the right and left hand sides of (1a) and (4), one can show that $\bar{x}$ satisfies the following Mean Field equation system (MF)

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} - \frac{1}{r}B^2 n \\ -\dot{n} &= A'n + q(1-\alpha)\bar{x} - qK(p_2)y \end{aligned} \tag{6}$$

with $\bar{x}(0) = \mu_0$, $n(T) = M(\bar{x}(T) - p_\lambda)$, $\lambda = P_0\left(D(\bar{x},y)\right)$, and $p_\lambda = \lambda p_1 + (1-\lambda)p_2$. Note that λ is the fraction of minor agents that goes towards p_1.

Assumption 1: The following Riccati equation has a unique solution:

$$\dot{\Pi} + \Pi A + A'\Pi - \frac{1}{r}\Pi B^2\Pi + q(1-\alpha)I_n = 0, \tag{7}$$

with $\Pi(T) = MI_n$.

Note that if $\alpha \le 1$, then (7) has a unique solution [18, page 23]. For more details about the the existence and uniqueness of solutions of (7), one can refer to [19]. Denoting Φ as the STM of $A - \frac{1}{r}B^2\Pi$, define the following entities:

$$\begin{aligned} R(t) &= \Phi(t,0) \\ \bar{R}(t) &= \frac{M}{r}\int_0^t \Phi(t,\sigma)B^2\Phi(T,\sigma)'\mathrm{d}\sigma \\ \Xi(y)(t) &= -\frac{q}{r}\int_0^t \int_T^\sigma \Phi(t,\sigma)B^2\Phi(\tau,\sigma)'K(p_2)y(\tau)\mathrm{d}\tau\mathrm{d}\sigma \end{aligned} \tag{8}$$

$$\begin{aligned} F(\lambda,y) &= P_0(H^\lambda(y)) \\ H^\lambda(y) &= \left\{x_0 \in \mathbb{R}^n | \beta' x_0 \le \delta + \Delta\left(K(p_2)y\right) + \alpha\Delta\left(R\mu_0 + \bar{R}p_\lambda + \Xi(y)\right)\right\}. \end{aligned} \tag{9}$$

In the following lemma, we show that there exists a one to one map between the fixed point paths $\bar{x}$ and the fixed points of the finite dimensional function $F(.,y)$. The existence of the latter is guaranteed under the following assumption.

Assumption 2: We assume that P_0 is such that the P_0-measure of hyperplanes is zero.

Using techniques similar to those used in [17, Theorem 6], one can show the following Lemma.

Lemma 1: Under Assumptions 1 and 2, the following statements hold:

1) $\bar{x}(t)$ is a solution of the MF equation system (6) if and only if it can be written under the form:
$$\bar{x}(t) = R(t)\mu_0 + \bar{R}(t)p_\lambda + \Xi(y)(t), \tag{10}$$
where $\lambda = F(\lambda, y)$ (i.e. λ is fixed point of $F(., y)$ defined in (9)).

2) The function $F(., y)$ has at least one fixed point λ (equivalently the MF equation system has at least one solution).

To prove the first point, we consider λ at first as a parameter. In this case, (6) is a linear forward-backward differential equation parameterized by λ. Under Assumption 7, n can be written as an affine function of $\bar{x}$, i.e. $n(t) = \Pi(t)\bar{x}(t) + \beta(t)$, where β is a well defined function. By replacing this form of n in (6), one can show that $\bar{x}$ is equal to (10). Thus, a fixed point path $\bar{x}$ is of the form (10), where $\lambda = P_0(D(\bar{x}, y)) = P_0(D(R(t)\mu_0 + \bar{R}(t)p_\lambda + \Xi(y)(t), y)) = P_0(H^\lambda(y)) = F(\lambda, y)$. Hence, λ is a fixed point of $F(., y)$, for fixed $y(t)$. The converse is proved by a simple verification argument.

By solving the backward equation in (6) and replacing the solution n in the forward equation, it can be shown as in [17] that the limiting macroscopic behaviors at the equilibrium $\bar{x}$ satisfy the following integro-differential equation
$$\dot{\bar{x}} = \mathcal{L}(\bar{x}, y)(t), \tag{11}$$
where $\bar{x}(0) = \mu_0$ and
$$\begin{aligned}\mathcal{L}(\bar{x}, y)(t) &= (A - \frac{1}{r}B^2\Pi)\bar{x} \\ &+ \frac{M}{r}B^2\Phi(T,t)'\bar{F} \circ \Delta\big(\alpha\bar{x} + K(p_2)y)\big)(p_1 - p_2) \\ &+ \frac{M}{r}B^2\Phi(T,t)'p_2 - \frac{q}{r}B^2\int_T^t \Phi(\sigma,t)'K(p_2)y(\sigma)\mathrm{d}\sigma \\ \bar{F}(s) &= P\Big(\beta' x_0 \leq \delta + s\Big).\end{aligned} \tag{12}$$

Using techniques similar to those used in [17, Theorem 9], one can show that in the case of a finite population of N agents, the strategy profile (u_i^*, u_{-i}^*), defined in (4) for any DA strategy v, is an $\epsilon-$Nash equilibrium with respect to the costs J_i, $i = 1, \ldots, N$ defined in (2a), where ϵ goes to zero as N increases to infinity.

To compute its optimal strategy, the DA should be able, for each strategy v, to anticipate *uniquely* the mA Nash equilibrium. In the following, we define a condition under which the uniqueness of the mA equilibria holds.

Assumption 3: We assume that $\bar{F}$ is differentiable with respect to λ and $|dF/d\lambda|(\lambda, y) < 1$ for all $\lambda \in [0, 1]$ and $y \in L^2([0, T])$.

Note that Assumption 3 can be satisfied if the initial spread of the minor agents is sufficient. In fact, $dF/d\lambda = \alpha\Delta\big(\bar{R}(p_1 - p_2)\big)d\bar{F}/ds$ (here the linear form Δ defined in (5) acts on the function $\bar{R}(t)(p_1 - p_2)$). If the probability density function $d\bar{F}/ds$ of βx_0 is strictly bounded by $1/|\alpha\Delta\big(\bar{R}(p_1 - p_2)\big)|$, then Assumption 3 is satisfied.

Example 2: If x_0 has a Gaussian distribution $\mathcal{N}(\mu_0, \Sigma_0)$ with $2\pi\beta'\Sigma_0\beta > \Big(\alpha\Delta\big(\bar{R}(p_1 - p_2)\big)\Big)^2$, then Assumption 3 is satisfied.

Under Assumption 3, the function $F(\lambda, y) - \lambda$ has a negative derivative w.r.t. λ. Therefore, one can state the following theorem.

Theorem 2: Under Assumptions 1, 2, and 3, given the DA strategy v, $F(., y)$ has a unique fixed point λ. Thus, the mA limiting game admits a unique Nash equilibrium. Moreover, the mA macroscopic behavior $\bar{x}$ satisfies the integro-differential equation (11).

B. DA Optimal Control Problem

Under Assumptions 1, 2, and 3 and given the DA strategy v, the unique mA Nash equilibrium is fully determined by (11). The DA optimal control problem is then defined as follows:
$$\begin{aligned}&\min_{v \in L_2([0,T])} \bar{J}_0\Big(v, \bar{x}(v)\Big) \\ &\text{s.t. } \dot{y} = A_0 y + B_0 v \text{ and } \dot{\bar{x}} = \mathcal{L}(\bar{x}, y)(t).\end{aligned} \tag{P1}$$

Because of the linear form Δ defined in (5), the dynamic constraint (11) is a nonlinear integro-differential equation, where the vector $\dot{\bar{x}}$ at time t depends on the entire path $\bar{x}(\sigma)$, $\sigma \in [0, T]$. Therefore, (P1) is a nonstandard optimal control problem, and the standard dynamic programming method cannot be applied. We solve the problem via the calculus of variations method [20]. We start at first by proving the existence of an optimal control law v^*.

Theorem 3: Under Assumptions 1, 2 and 3, the DP optimal control problem (P1) has an optimal control law v^*.

Having proved the existence of an optimal control law, we now characterize the optimal solution v^*, as the solution of a certain differential equation, see Theorem 4 below. In the following, we denote by v^* the optimal control law, y^* the optimal DA state and $\bar{x}^*$ the corresponding mA optimal mean field. The idea of the following analysis is to derive a first variation condition on the cost functional in (P1) by considering a perturbation $v = v^* + \eta\delta v$, where $\eta \in \mathbb{R}$, and $\delta v \in L_2([0, T])$. This condition is, as it is shown below (16a)-(16b)-(17), a Backward functional ordinary differential equation (BODE) involving the optimal control law v^*. We start by computing the Gâteaux derivatives [20] of y and $\bar{x}$ at v^* in the direction δv:
$$\begin{aligned}\frac{d}{d\eta}y(v^* + \eta\delta v)\Big|_{\eta=0} &:= \delta y \\ \frac{d}{d\eta}\bar{x}(v^* + \eta\delta v)\Big|_{\eta=0} &:= \delta\bar{x},\end{aligned} \tag{13}$$
where,
$$\begin{aligned}&\frac{d}{dt}\delta y = A_0\delta y + B_0\delta v, \quad \delta y(0) = 0 \\ &\frac{d}{dt}\delta\bar{x} = \mathcal{L}_1(\delta y)(t) + \mathcal{L}_2(\delta\bar{x})(t), \quad \delta\bar{x}(0) = 0,\end{aligned}$$

and $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) is a continuous linear operator from the Hilbert space $L^2([0,T],\mathbb{R}^{n_1})$ (resp. $L^2([0,T],\mathbb{R}^n)$) to $L^2([0,T],\mathbb{R}^n)$ (resp. $L^2([0,T],\mathbb{R}^n)$, such that for all $z_1 \in L^2([0,T],\mathbb{R}^{n_1})$ and $z_2 \in L^2([0,T],\mathbb{R}^n)$,

$$\begin{aligned}\mathcal{L}_1(z_1)(t) &= -\frac{q}{r}B^2\int_T^t \Phi(\sigma,t)'K(p_2)z_1(\sigma)\mathrm{d}\sigma \\ &\quad + \frac{M}{r}\xi^*\Delta\Big(K(p_2)z_1\Big)B^2\Phi(T,t)'(p_1-p_2) \\ \mathcal{L}_2(z_2)(t) &= (A-\frac{1}{r}B^2\Pi)z_2(t) \\ &\quad + \frac{M\alpha}{r}\xi^*\Delta\Big(z_2\Big)B^2\Phi(T,t)'(p_1-p_2) \\ \xi^* &= \frac{d\bar{F}}{ds}\Big(\Delta\big(\alpha\bar{x}^*+K(p_2)y^*\big)\Big). \end{aligned} \quad (14)$$

To extract the Gâteaux derivatives δy and $\delta\bar{x}$ from the expressions of $\mathcal{L}_1$ and $\mathcal{L}_2$, we need to compute the relevant adjoint operators. We recall from [21] that the adjoint operator of a linear continuous operator $\mathcal{G}$ defined on a Hilbert space H_1 is the linear continuous operator $\mathcal{G}^*$ defined on a Hilbert space H_2 satisfying for all $x \in H_1$ and $y \in H_2$ $\langle\mathcal{G}(x),y\rangle = \langle x,\mathcal{G}^*(y)\rangle$. Using Fubini-Tonelli's theorem [22], one can show that the adjoint operators of $\mathcal{L}_1$ and $\mathcal{L}_2$ are respectively

$$\begin{aligned}\mathcal{L}_1^*(z)(t) &= \frac{q}{r}K(p_2)'\int_0^t \Phi(t,\sigma)B^2z(\sigma)\mathrm{d}\sigma \\ &\quad + \xi^*K(p_2)'H(t)\int_0^T \Phi(T,\sigma)B^2z(\sigma)\mathrm{d}\sigma \\ \mathcal{L}_2^*(z)(t) &= (A-\frac{1}{r}B^2\Pi)'z(t) \\ &\quad + \xi^*\alpha H(t)\int_0^T \Phi(T,\sigma)B^2z(\sigma)\mathrm{d}\sigma,\end{aligned} \quad (15)$$

for all $z \in L^2([0,T],\mathbb{R}^n)$, where

$$H(t) = \frac{M^2q}{r^2}\int_0^t \phi(\eta,t)'B^2\phi(\eta,T)\mathrm{d}\eta(p_1-p_2)^2.$$

Given an optimal control law v^*, we define the following BODE:

$$-\dot{P} = A_0'P + \mathcal{L}_1^*(Q)(t) \quad (16a)$$

$$-\dot{Q} = \mathcal{L}_2^*(Q)(t) \quad (16b)$$

with $P(T)=0$ and $Q(T)=M_0(\bar{x}^*(T)-p_2)$. We now state the main result of this paper.

Theorem 4: Under Assumptions 1, 2, and 3, if v^* is an optimal control law of (P1) and the corresponding BODE (16a)-(16b) has a unique solution (P,Q), then

$$v^* = -\frac{1}{r_0}B_0'P. \quad (17)$$

In the following, we study the existence and uniqueness of solutions of (16a)-(16b). Given the function Q, equation (16a) is a linear Ordinary Differential Equation (ODE) which has a unique solution. Thus, it is sufficient to study the second equation (16b). We define the matrix

$$\Sigma = \alpha\int_0^T\int_\sigma^T \Big(\Phi(T,\sigma)B\Big)^2\Phi(\tau,T)'H(\tau)\mathrm{d}\tau\mathrm{d}\sigma. \quad (18)$$

Assumption 4: Either ξ^* is equal to zero or $1/\xi^*$ is not an eigenvalue of Σ, where ξ^* is defined in (14).

Assumption 4 can be satisfied, for example, in the following two cases:

1) If the initial spread of the agents is sufficient ($d\bar{F}/ds$ is low enough).
2) If $d\bar{F}/ds$ is bounded, and T is small enough.

In fact, $\xi^*\Sigma$ is in both cases negligible with respect to I_n. Hence, $1/\xi^*$ is not an eigenvalue of Σ.

Lemma 5: Under Assumption 4, (16b) has a unique solution.

Theorem 3 asserts that the DA can always act optimally. Multiple optimal control laws v^* may exist, each characterized by (16a)-(16b)-(17). To compute its optimal strategy, the DA should solve the coupled nonlinear forward-backward functional differential equations (1b)-(11)-(16a)-(16b), where v in (1b) is equal to (17). Once the DA plays v_*, the mA act subsequently as follows. Each mA computes the unique fixed point of $F(,.y^*)$, the corresponding macroscopic behavior (10) and its best response (4) to this behavior.

In view of the nonlinear functional (14) of $\bar{x}^*$ and y^*, finding an explicit solution of (1b)-(11)-(16a)-(16b) is non trivial. Noting that $d\bar{F}/ds$ is the probability density function of $\beta'x_0-\delta$ (see (12)), one can hope to compute an explicit solution in case this random variable is uniformly distributed, i.e. $d\bar{F}/ds$ is piecewise constant. In the following section, we investigate this case.

IV. Case of Uniform Initial Distribution

In this section, we study a special case where the minor agents' initial conditions are uniformly distributed on a line segment. More precisely, we assume that $\beta'x_0-\delta$ has a uniform distribution $U([-a_1-c/2,a_2+c/2])$, where $a_1>0$, $a_2>0$, and $c>0$. We show in this case that if the initial spread of the minor agents is sufficient (see Assumption 5 below), then there exists a unique Stackelberg solution. There are two distinct destination points and we describe the way the population of minor agents splits between the two.

The function $\bar{F}$ is piecewise differentiable. Therefore, we need an alternative to Assumption 3, under which the uniqueness of the mA Nash equilibria holds. Moreover, in order to apply the variational methods of Subsection III-B, we require $\bar{F}$ to stay in a differentiable domain for all the DA strategies, which is the case when the mA agents are spread enough (see Lemma 7 below).

Assumption 5: We assume that $c > M\alpha\Big|\Delta\big(\bar{R}(p_1-p_2)\big)\Big|$.

Under Assumption 5, given the DA strategy v, the mA limiting game admits a unique Nash equilibrium by virtue of Theorem 2.

Theorem 6: Under Assumptions 1, 2 and 5, the DA optimal control problem (P1) has an optimal control law v^*.

Lemma 7: Under Assumptions 1, 2 and 5, there exists $c_0>0$ independent of v such that for all $c>c_0$, there exists a unique mA Nash equilibrium corresponding to $\lambda\in(0,1)$.

For the rest of the analysis, we assume that $c>c_0$. In this case, the unique fixed point λ corresponding to a DA optimal control law v^* is in $(0,1)$. Since F is differentiable in $(0,1)$, one can use techniques similar to those used in Theorem 4 to show that v_2^* satisfies (17) (provided that the Assumptions 1, 2 and 5 are satisfied, and $1/c$ is not an eigenvalue of Σ defined in (18)).

In the following, we characterize the Stackelberg solution, that is, the DA optimal state y^* and the corresponding mA macroscopic behavior captured here by $\bar{x}^*$. The pair $(\bar{x}^*, y^*)$ satisfies (1b)-(6)-(16a)-(16b)-(17). We define the states $h = (\bar{x}^*, y^*, q_1)$, $d = (n, P, Q, q_2)$, where $q_1(t) = \int_0^t \Phi(T,\sigma)B^2 Q(\sigma)\mathrm{d}\sigma$ and $q_2(t) = \int_t^T \Phi(T,\sigma)B^2 Q(\sigma)\mathrm{d}\sigma$. (h,d) satisfies

$$\begin{aligned}\dot{h} &= K_1(t)h + K_2(t)d \\ \dot{d} &= K_3(t)h + K_4(t)d\end{aligned} \tag{19}$$

with $h(0) = h_0 = (\mu_0, y^0, 0)$ and $d(T) = K_5 h(T) + K_\lambda$, where $K_1(t) = \mathrm{diag}(A, A_0, 0)$,

$$K_2(t) = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \quad K_5 = \begin{bmatrix} MI_n & 0 & 0 \\ 0 & 0 & 0 \\ M_0 I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$K_3(t) = \begin{bmatrix} -q(1-\alpha)I_n & qK(p_2) & 0 \\ 0 & 0 & k_4 \\ 0 & 0 & -\alpha H(t)/c \\ 0 & 0 & 0 \end{bmatrix},$$

$$K_4(t) = \begin{bmatrix} -A' & 0 & 0 & 0 \\ 0 & -A_0' & 0 & -K(p_2)'H(t)/c \\ 0 & 0 & k_5 & -\alpha H(t)/c \\ 0 & 0 & -k_3 & 0 \end{bmatrix},$$

$K_\lambda = -\Big(Mp_\lambda, 0, M_0 p_2, 0\Big)$, $k_1 = -\frac{1}{r}B^2$, $k_2 = -\frac{1}{r_0}B_0^2$, $k_3 = \Phi(T,t)B^2$, $k_4 = -K(p_2)'\Big(\frac{q}{r}\Phi(t,T) + \frac{H(t)}{c}\Big)$ and $k_5 = -(A - \frac{1}{r}B^2\Pi)'$.

The equation system (19) is a system of coupled nonlinear Forward-Backward differential equations. In fact, the final condition $d(T)$ depends through λ non-linearly on the path $(\bar{x}^*(\sigma), y^*(\sigma))$, $\sigma \in [0,T]$. To solve this system, we start by considering λ as a parameter. The equation system is then a system of coupled Linear Forward-Backward ordinary differential equations parameterized by λ that we denote λ-LFBODE. Under Assumption 6 below, one can solve the λ-LFBODE, and give an explicit solution parameterized by λ. By replacing the parameterized solutions in the expression of $\lambda = P_0(D(\bar{x}^*, y^*))$, we construct a one-to-one map between the solutions of (19) and the fixed points of a finite dimensional operator acting on λ. Thus, we start by considering λ in K_λ as a parameter.

Assumption 6: The following generalized Riccati equation has a unique solution

$$\dot{W} = K_4 W - W K_1 - W K_2 W + K_3, \tag{20}$$

with $W(T) = K_5$.

For more details about the the existence and uniqueness of solutions of (20), one can refer to [19].

Theorem 8: Under Assumption 6, the λ-LFBODE has a unique solution (h,d). Moreover, $d = Wh + S$, where S is the unique solution of

$$\dot{S} = (K_4 - WK_2)S, \quad S(T) = K_\lambda. \tag{21}$$

We consider respectively Φ_1 and Φ_2 the STM of $K_1 + K_2 W$ and $K_4 - WK_2$, and we define

$$\begin{aligned} F_u(\lambda) &= P_0(H^\lambda) \\ H^\lambda &= \Big\{x_0 \in \mathbb{R}^n | \beta' x_0 \leq \delta + \\ &\quad \Delta\Big(\Big(\alpha I_n, K(p_2), 0\Big)'\Big(R_1 h_0 + R_2 K_\lambda\Big)\Big)\Big\} \\ R_1(t) &= \Phi_1(t,0) \\ R_2(t) &= \int_0^t \Phi_1(t,\sigma)K_2(\sigma)\Phi_2(\sigma,T)\mathrm{d}\sigma. \end{aligned} \tag{22}$$

Using techniques similar to those used in Lemma 1, one can show the following result.

Theorem 9: Under Assumptions 1, 2, 5, and 6, (19) has a unique solution (h,d), where $d = Wh + S$ and

$$h(t) = R_1(t)h_0 + R_2(t)K_\lambda, \tag{23}$$

where λ is the unique fixed point of F_u in (22).

Theorem 9 describes the unique way the limiting population splits between the destination points under the social and advertisement effects. In fact, the fixed point λ is the fraction of minor agents that chooses p_1. One can apply the bisection method to find λ. Once λ is computed, the agents can compute the vectors $(h, d = Wh + S)$, where h is given by (23). The dominating player can then compute its optimal strategy (17), where P is the second component of d. Furthermore, the minor players can predict their limiting macroscopic behavior, the first component of d, and compute their optimal strategies (4). The minor agents' strategy profile is an $\epsilon-$Nash equilibrium with respect to the minor players costs.

V. Simulations

To illustrate the collective choice mechanism in the presence of social and advertisement effects, we consider a group of 6000 agents initially uniformly distributed on the segment $[-25, 5]$. The agents are choosing between $p_1 = -20$ and $p_2 = 20$. The social effect is represented by $\alpha\bar{x}$, where $\alpha = 0.5$. We consider two scenarios. In the first one, the agents make their choices in the absence of an advertisement effect ($K(p_2) = 0$), while in the second scenario, a dominating agent advertises for p_2. The advertisement effect is modeled in the cost by $K(p_2)y = p_2 y$, where y is the (influence) state of the dominant agent. We set $T = 3s$, $A = 0.5$, $B = 0.5$, $A_0 = -0.1$, $B_0 = 0.1$, $y_0 = 0$, $q = 10$, $r = r_0 = 10$, and $M = M_0 = 2000$. In the absence of an advertisement effect, $\lambda = 0.84$ is the unique fixed point of F_u defined in (22). Accordingly, 84% of the minor agents go towards p_1 (Fig. 1). On the other hand, the presence of advertisement for alternative p_2 increases from 16% to 87% the fraction of minor agents that go towards p_2 (Fig. 2).

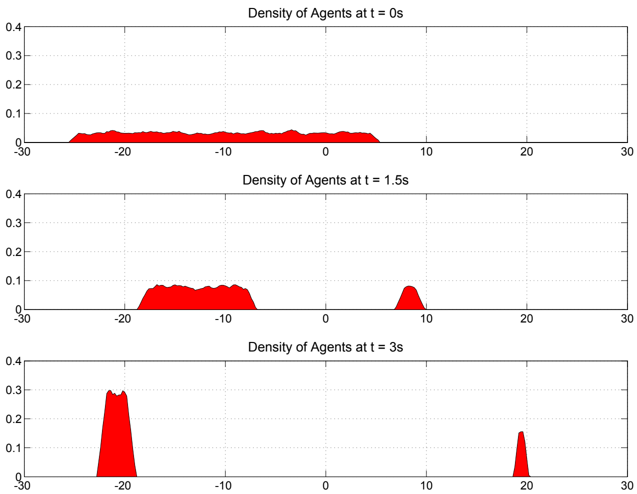

Fig. 1. Evolution of the density of minor players in the absence of advertisement effect - 16% of the minor agents go towards $p_2 = 20$. In the absence of an advertisement effect, the majority of the population goes towards p_1.

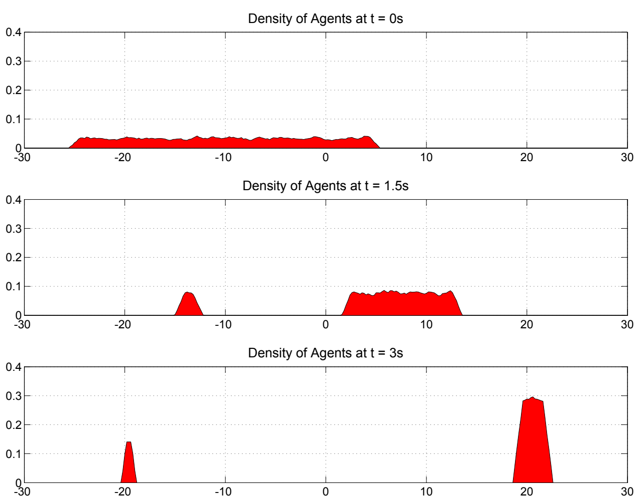

Fig. 2. Evolution of the density of minor players in the presence of advertisement effect - 87% of the minor agents go towards $p_2 = 20$. Under the advertisement effect, the majority of the population goes towards p_2.

VI. Conclusion

We introduce in this paper a dynamic collective choice model in the presence of social and advertisement effects. In this model, a large group of minor consumers choose between two alternatives while influenced by their average and an advertisement effect. The latter is exerted by a Stackelbergian dominating advertiser aiming at convincing the population of minor agents to choose p_2. We consider the limiting infinite population game and derive conditions under which a Stackelberg solution exists. In case of minor agents initially distributed uniformly on a line segment, we characterize the solutions by a vector $(\lambda, 1-\lambda)$ describing the way the population of consumers splits between the alternatives. The scalar λ is a fixed point of a well defined finite dimensional operator. Finally, it is of interest for future work to extend the results to the case of multiple competitive advertisers, with multiple potential choices.

References

[1] F. Koppelman and V. Sathi, "Incorporating variance and covariance heterogeneity in the generalized nested logit model: an application to modeling long distance travel choice behavior," *Transportation Research*, vol. 39, pp. 825–853, 2005.

[2] C. Bhat and J. Guo, "A mixed spatially correlated logit model: formulation and application to residential choice modeling," *Transportation Research*, vol. 38, pp. 147–168, 2004.

[3] R. Nakajima, "Measuring peer effects on youth smoking behaviour," *The Review of Economic Studies*, vol. 74, no. 3, pp. 897–935, 2007.

[4] G. M. Erickson, "Differential game models of advertising competition," *European Journal of Operational Research*, vol. 83, no. 3, pp. 431–438, 1995.

[5] T. Basar and G. J. Olsder, *Dynamic noncooperative game theory*. Siam, 1999, vol. 23.

[6] M. Huang, R. P. Malhamé, and P. E. Caines, "Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle," *Communications in Information & Systems*, vol. 6, no. 3, pp. 221–252, 2006.

[7] M. Huang, P. E. Caines, and R. P. Malhamé, "Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized epsilon-Nash equilibria," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1560–1571, 2007.

[8] ——, "Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, 2003, pp. 98–103.

[9] J. M. Lasry and P. L. Lions, "Jeux à champ moyen. I–le cas stationnaire," *Comptes Rendus Mathématique*, vol. 343, no. 9, pp. 619–625, 2006.

[10] ——, "Jeux à champ moyen. II–horizon fini et contrôle optimal," *Comptes Rendus Mathématique*, vol. 343, no. 10, pp. 679–684, 2006.

[11] ——, "Mean field games," *Japanese Journal of Mathematics*, vol. 2, pp. 229–260, 2007.

[12] A. Bensoussan, M. Chau, and P. Yam, "Mean field games with a dominating player," *arXiv preprint arXiv:1404.4148*, 2014.

[13] A. Bensoussan, M. Chau, and S. Yam, "Mean field stackelberg games: Aggregation of delayed instructions," *SIAM Journal on Control and Optimization*, vol. 53, no. 4, pp. 2237–2266, 2015.

[14] H. Von Stackelberg, *Market structure and equilibrium*. Springer Science & Business Media, 2010.

[15] M. Huang, "Large-population LQG games involving a major player: the Nash certainty equivalence principle," *SIAM Journal on Control and Optimization*, vol. 48, no. 5, pp. 3318–3353, 2010.

[16] R. Salhab, R. P. Malhamé, and J. Le Ny, "A dynamic collective choice model with an advertiser," *Cahiers du GERAD:G-2016-46*, 2016.

[17] ——, "A dynamic game model of collective choice in multi-agent systems," *arXiv preprint arXiv:1506.09210*, 2015 (Submitted For Publication).

[18] B. D. Anderson and J. B. Moore, *Optimal control: linear quadratic methods*. Courier Dover Publications, 2007.

[19] G. Freiling, "A survey of nonsymmetric Riccati equations," *Linear Algebra and its Applications*, vol. 351, pp. 243–270, 2002.

[20] F. Clarke, *Functional analysis, calculus of variations and optimal control*. Springer Science & Business Media, 2013, vol. 264.

[21] W. Rudin, *Functional Analysis*, ser. International series in pure and applied mathematics. McGraw-Hill, 1991.

[22] ——, *Real and Complex Analysis*, 3rd ed. McGraw-Hill Inc., 1987.