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On the Dubins Traveling Salesman Problem

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***Abstract*—We study the traveling salesman problem for a Dubins vehicle. We prove that this problem is NP-hard, and provide lower bounds on the approximation ratio achievable by some recently proposed heuristics. We also describe new algorithms for this problem based on heading discretization, and evaluate their performance numerically.**

***Index Terms*—Algorithms, motion planning, traveling salesman problem (TSP), unmanned aerial vehicles (UAVs).**

I. Introduction

In an instance of the Traveling Salesman Problem (TSP), we are given the distances between any pair of n points. The problem is to find the shortest closed path (tour) visiting every point exactly once. We also call this problem the tour-TSP to distinguish it from the path-TSP, where the requirement that the vehicle must start and end at the same point is removed. This famously intractable problem is often encountered in robotics, and has traditionally been solved in two steps within the common layered controller architectures for mobile robots. At the higher decision-making level, the dynamics of the robot are usually not taken into account and the mission planner might typically chose to solve the TSP for the Euclidean metric (ETSP), i.e., using the Euclidean distances between waypoints. For this purpose, one can directly exploit many existing results on the ETSP (or more general TSPs on graphs), see e.g., [1], [2]. This first step determines the order in which the waypoints should be visited by the robot. At the lower level, a path planner takes as an input this waypoint ordering, and designs feasible trajectories between the waypoints respecting the dynamics of the robot. In this technical note, we assume that the robot has a limited turning radius and can be modeled as a Dubins vehicle [3], [4]. Consequently, the path planner could solve a sequence of Dubins Shortest Path Problems (DSPP) between the successive waypoints. DSPPs have also been extensively studied since the work of Dubins [3], most of the literature concentrating on designing shortest paths between an initial and final configuration for a Dubins vehicle moving among obstacles, see e.g., [4]–[9]. In fact, one should also consider shortest Dubins paths through a sequence of ordered waypoints of length greater than two, since the vehicle configuration at an intermediate waypoint influences the length of the shortest Dubins path between the next two waypoints. This problem was studied by [10], [11] for an environment without obstacles.

Even if each problem is solved optimally however, the ad-hoc separation into two successive steps can be inefficient, since the sequence of points chosen by the TSP algorithm is often hard to follow for the physical system. In order to improve the performance of unmanned aerial systems in particular, researchers are now working on integrating the mission planning and path planning stages [12], [14]. In this note we consider the TSP for the Dubins vehicle (DTSP), in a *planar environment without obstacles*, a problem introduced by Savla *et al.* in [17]. The Dubins model provides a good kinematic model for fixed wing aircraft. At the same time, we can quickly compute the length of the shortest path between any two configurations of the Dubins vehicle, a necessary building block to design algorithms with good performance for the DTSP.

A stochastic version of the DTSP for which the points are distributed randomly and uniformly in the plane was considered in [13]–[16]. Here however, we focus on algorithms and worst-case bounds for the more standard problem where no probability distribution is given for the input. In that case, most of the recently proposed algorithms seem to build on a preliminary solution obtained for the ETSP [11], [17]–[19]. More detailed references on the DTSP can be found in [14].

1) Contributions of This Work: In this note, we first prove that the DTSP is NP-hard, thus justifying the work on heuristics and algorithms that approximate the optimal solution. Recall that an α-approximation algorithm (with *approximation ratio* $\alpha \geq 1$) for a minimization problem is an algorithm that produces *on any instance of the problem* with optimum OPT, a feasible solution whose value Z is within a factor α of the optimum, i.e., such that $OPT \leq Z \leq \alpha\, OPT$. In general, α is allowed to depend on the input parameters of the problem, such as the number of points in the DTSP. This definition of approximation ratio, used throughout the technical note, corresponds to the worst-case performance of the algorithm [20]. On the negative side, we give some lower bounds on the approximation ratio achievable by recently proposed heuristics. Following a tour based on the ETSP or-

Manuscript received June 14, 2008; revised March 11, 2009; accepted February 03, 2010. Date of publication August 30, 2011; date of current version December 29, 2011. This work was supported by Air Force—DARPA—MURI award 009628-001-03-132 and Navy ONR award N00014-03-1-0171. Recommended by Associate Editor E. Fabre.

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Digital Object Identifier 10.1109/TAC.2011.2166311

dering or the ordering of Tang and Özgüner [12] cannot achieve an approximation ratio better than $\Omega(n)$[1]. The same is true for the nearest neighbor heuristic, in contrast to the ETSP where it achieves a $O(\log n)$ approximation [21]. Then we propose an algorithm based on heading discretization, which is a standard technique in the work on curvature-constrained shortest path problems [5]. Its theoretical performance does not improve on the previously mentioned heuristics. However, numerical simulations show a significant performance improvement in randomly generated instances over other heuristics when the inter waypoint distances are smaller than the turning radius of the vehicle.

The rest of this technical note is organized as follows. We recall some facts about Dubins paths in Section II and reduce the DTSP to a finite dimensional optimization problem. In Section III we show that the DTSP is NP-hard. In Section IV-A we provide lower bounds on the approximation ratios of various recently published heuristics. Section IV-B describes our algorithms based on heading discretization. They return in time $O(n^3)$ a tour within $O\left(\min\left((1+\rho/\epsilon)\log n, (1+\rho/\epsilon)^2\right)\right)$ of the optimum, where ρ is the minimum turning radius of the vehicle and ϵ is the minimum Euclidean distance between any two waypoints. Note that throughout the technical note, we fix ρ but ϵ is allowed to depend on the problem instance. In particular if the waypoints are sampled in a compact environment we have necessarily $\epsilon = O(1/\sqrt{n})$. Finally, Section V discusses the results of our numerical simulations.

II. Problem Formulation

A Dubins vehicle in the plane has its configuration described by its position and heading $(x, y, \theta) \in \mathbb{R}^2 \times S^1$. Its equations of motion are

$$\dot{x} = v_0 \cos(\theta),$$
$$\dot{y} = v_0 \sin(\theta),$$
$$\dot{\theta} = \frac{v_0}{\rho} u, \quad \text{with } u \in [-1, 1]$$

where ρ is the minimum turning radius of the vehicle, and u is the available control. Without loss of generality, we assume that the speed v_0 of the vehicle is normalized to 1. Dubins [3] characterized curvature-constrained shortest paths between an initial and a final configuration. Let P be a feasible path. We call a nonempty subpath of P a C-segment or an S-segment if it is a circular arc of radius ρ or a straight line segment, respectively.

Theorem 1 (Dubins [3]): A shortest path between any two configurations of a Dubins vehicle in an environment without obstacles is of type CCC or CSC, or a subpath of a path of either of these two types.

We refer to these minimal-length paths as Dubins paths. When a subpath is a C-segment, it can be a left or a right hand turn: denote these two types of C-segments by L and R respectively. The DTSP asks, for a given set of points in the plane, to find the shortest tour through these points that is feasible for a Dubins vehicle. By Theorem 1, the minimum length path between an initial and a final configuration can be found among the six paths $\{LSL, RSR, RSL, LSR, RLR, LRL\}$. Each of these paths can be explicitly computed and therefore finding the optimum path and length between any two configurations can be done in constant time [3]. Solving the DTSP reduces then to choosing a permutation of the points specifying in which order to visit them, as well as choosing a heading for the vehicle at each of these points.

III. Complexity of the DTSP

It is usually accepted that the DTSP is NP-hard and the goal of this section is to prove this claim rigorously. Note that adding the curvature constraint to the Euclidean TSP could well make the problem easier, as in the bitonic TSP [22, p. 364] for example. Hence, the statement does not follow trivially from the NP-hardness of the ETSP [23], [24]. In the proof of Theorem 2, we consider, without loss of generality, the decision version of the problem, which we also call DTSP. That is, given a set of points in the plane and a number $L > 0$, DTSP asks if there exists a tour for the Dubins vehicle visiting all these points exactly once, of length at most L.

Theorem 2: Tour-DTSP and path-DTSP are NP-hard.

Proof: This is a corollary of Papadimitriou's proof of the NP-hardness of ETSP, to which we refer [23]. First recall the Exact Cover Problem: given a family F of subsets of the finite set U, is there a subfamily F' of F, consisting of disjoint sets, such that F' covers U? This problem is known to be NP-complete [25]. Papadimitriou described a polynomial-time reduction of Exact Cover to ETSP. That is, given an instance of the Exact Cover problem, we can construct an instance of the Euclidean Traveling Salesman Problem and a number L such that the Exact Cover problem has a solution if and only if the ETSP has an optimal tour of length less than or equal to L. The important fact to observe however, is that if Exact Cover does not have a solution, Papadimitriou's construction gives an instance of the ETSP that has an optimal tour of length $\geq (L+\delta)$, for some $\delta > 0$, and not just $>L$. More precisely, letting $a = 20$ exactly as in his proof, we can take $0 < \delta < \sqrt{a^2+1} - a$.

Now from [17], there is a constant C such that for any instance $\mathcal{P}$ of ETSP with n points and length $ETSP(\mathcal{P})$, the optimal DTSP tour for this instance has length less than or equal to $ETSP(\mathcal{P}) + Cn$. Then if we have n points in the instance of the ETSP constructed as in Papadimitriou's proof, we simply rescale all the distances by a factor $2Cn/\delta$. If Exact Cover has a solution, the ETSP instance has an optimal tour of length no more than $2CnL/\delta$ and so the curvature constrained tour has a length of no more than $2CnL/\delta + Cn$. If Exact Cover does not have a solution, the ETSP instance has an optimal tour of length at least $2CnL/\delta + 2Cn$, and the curvature constrained tour as well. So Papadimitriou's construction, rescaled by $2Cn/\delta$ and using $2CnL/\delta + Cn$ instead of L, where n is the number of points used in the construction, provides a reduction from Exact Cover to DTSP. ■

IV. Approximation Algorithms

A. Hard Instances for Previously Proposed Algorithms

If the Euclidean distances between the waypoints to visit are large with respect to ρ, the DTSP and ETSP behave similarly (see, e.g., [14]). Accordingly, researchers have tried in previous work to apply to the DTSP the waypoint ordering optimal for the ETSP [11], [17], [18], and have concentrated on the choice of headings. Theorem 3 provides a limit on the performance one can achieve using this approach, which becomes particularly significant when the points are densely distributed with respect to ρ. We also describe two heuristics for the DTSP that are not based on the optimal ETSP ordering. The *nearest neighbor heuristic* produces a complete solution for the DTSP, including a waypoint ordering and a heading at each point. We start with an arbitrary point, and choose its heading arbitrarily, fixing an initial configuration. Then at each step, we find a point which is not yet on the path but closest to the last added configuration according to the Dubins metric. This is possible since we also have a complete characterization of the Dubins distance and path between an initial configuration and a final point with free heading [8]. We add this closest point to the path with the associated optimal arrival heading. When all nodes have been added to the path, we add a Dubins path connecting the last obtained configuration and the initial configuration. Note that it is known that the nearest neighbor heuristic achieves a $O(\log n)$ approximation ratio for the ETSP, which is a particular case of the symmetric TSP [21]. The

[1]We say $f(n) = O(g(n))$ if there exists $c > 0$ such that $f(n) \leq cg(n)$ for all n, and $f(n) = \Omega(g(n))$ if there exists $c > 0$ such that $f(n) \geq cg(n)$ for all n.

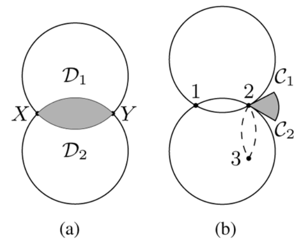


Fig. 1. A path between X and Y contained in the shaded region $\mathcal{D}_1 \cap \mathcal{D}_2$ is called a direct path. On Fig. 1(b), the vehicle cannot pass through points 1, 2 and 3 using only direct paths. For direct paths $1 \rightarrow 2$, we show the range of possible final headings at 2, delimited by the tangent directions to $\mathcal{C}_1$ and $\mathcal{C}_2$. We also delimit the region of direct paths $2 \rightarrow 3$.

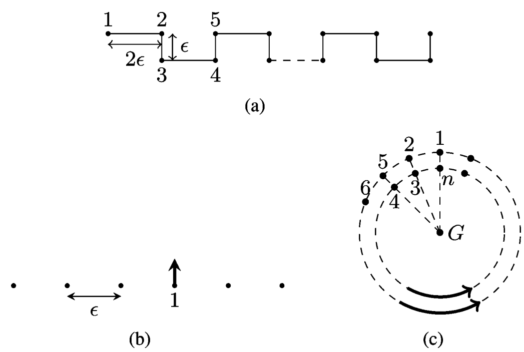


Fig. 2. Waypoint configurations that are hard instances for some proposed ordering methods. (a) ETSP ordering. (b) Nearest-neighbor heuristic. (c) Tang and Özgüner ordering.

second heuristic, due to Tang and Özgüner [12], only produces a waypoint ordering (in [12], the authors then produce locally optimal headings for this choice of waypoint ordering using a gradient descent algorithm). To construct this ordering, we find the geometric center G of the waypoints, and calculate the orientation angle of each waypoint with respect to G. We then sort the points by increasing values of their orientation to determine the traverse order. In the following theorem, for an instance $\mathcal{P}$ of the DTSP, we denote by $DTSP(\mathcal{P})$ the length of the optimal Dubins tour.

Theorem 3: There exist instances $\{\mathcal{P}_n\}_{n \geq 1}$ of the DTSP, where $\mathcal{P}_n$ has n points, and constants $K > 0$ and $n_0 \in \mathbb{N}$, such that any algorithm following the optimal ETSP ordering cannot approximate $DTSP(\mathcal{P}_n)$ within a factor better than Kn, for all $n \geq n_0$. This statement is also true for the nearest neighbor heuristic and the Tang and Özgüner ordering [12] (for possibly different instances and constants).

Proof: There are exactly two circles $\mathcal{C}_1, \mathcal{C}_2$ of radius ρ passing through two points X and Y in the plane with Euclidean distance $\|X - Y\| < 2\rho$, see Fig. 1(a). These circles define the boundaries of two closed discs $\mathcal{D}_1$ and $\mathcal{D}_2$. Following [10], we call a path from X to Y a *direct path* if it is contained in $\mathcal{D}_1 \cap \mathcal{D}_2$, and a *detour path* otherwise. It is shown in [10] that a Dubins path from X to Y is of length strictly smaller than $\pi\rho$ if and only if it is a direct path. Now consider the configuration of points shown in Fig. 2(a), with $\epsilon < \rho/\sqrt{2}$. Let n be the number of points, and suppose $n = 4m + 1$, m an integer. For clarity we focus on path-TSP (the extension to the tour-TSP case is easy, by adding a similar path in the reverse direction). The optimal Euclidean path-TSP is shown on Fig. 2(a) as well. Suppose now that a Dubins vehicle tries to follow the points in this order. Then for each sequence of 5 consecutive points the vehicle will have to execute at least two detour paths. For example, if the vehicle follows a direct path between points 1 and 2, it follows from a simple geometric argument that point 3 is in the disc $\mathcal{D}_2$ (see Fig. 1(b)). Moreover it is shown in [10] that the set of possible headings at 2 are directed outside of $\mathcal{D}_2$ as shown on Fig. 1(b), whereas the direct paths from 2 to 3 are contained in $\mathcal{D}_2$. Hence the path from 2 to 3 must be a detour path, i.e., of length greater than $\pi\rho$. The same argument shows that any direct path in the sequence, in the vertical or horizontal direction, must be followed by a detour path. Hence the length of a curvature constrained path through points 1 to 5 in this order is lower bounded by $2\epsilon + 2\pi\rho$. The length of a Dubins path following the ETSP ordering will then be greater than $(n - 1)(\epsilon + \pi\rho)/2$.

On the other hand, a Dubins vehicle can simply go through all the points on the top line, execute a U-turn of length at most $6\pi\rho$ (considering CCC paths [6, p.28]), and then go through the points on the lower line, providing an upper bound of $2\epsilon(n - 1) + 6\pi\rho$ for the optimal solution. We deduce that the worst case approximation ratio α_n of the algorithm for instances with n points is at least:

$$\alpha_n \geq \frac{(n-1)(\epsilon + \pi\rho)}{4(n-1)\epsilon + 12\pi\rho}. \tag{1}$$

In particular if we choose $\epsilon \leq 1/(n-1)$ for problem instances with n points, we get $\alpha_n = \Omega(n)$.

For the nearest-neighbor heuristic, we use the instances shown on Fig. 2(b), which includes the first configuration χ (waypoint and heading) chosen by the algorithm. Note that the C-segments starting from a configuration $\chi = (x, y, \theta)$ are circular arcs on one of the two circles of radius ρ tangent to the direction θ at (x, y), denoted $\mathcal{C}_L^\chi$ and $\mathcal{C}_R^\chi$ respectively, boundaries of the closed disks denoted $\mathcal{D}_L^\chi$ and $\mathcal{D}_R^\chi$. If ϵ is small enough, all the subsequent points are in the interior of the discs $\mathcal{D}_L^\chi$ or $\mathcal{D}_R^\chi$, hence at Dubins distance greater than $\pi\rho$ of the initial configuration χ (see [8]). The nearest neighbor heuristic chooses one of these points and as $\epsilon \rightarrow 0$, it reaches this chosen point by a Dubins path that tends to a circle of radius ρ and with a final heading that tends to the initial heading (in the limit $\epsilon \rightarrow 0$, a non trivial Dubins path from a point to itself is just a circle of radius ρ passing through the point). Overall it produces a path of length at least $n\pi\rho$ (in fact, this length tends to $n2\pi\rho$ as $\epsilon \rightarrow 0$), whereas there is clearly a (Dubins) straight path of length $C_1 + C_2 n\epsilon$ through these points, for some constants C_1 and C_2. Hence again $\alpha_n = \Omega(n)$.

Finally, for the Tang and Özgüner ordering, we close the path of Fig. 2(a) into a cycle to obtain the configuration of Fig. 2(c). The waypoints are on two concentric circles of radius R and $R - \epsilon$, with $R - \epsilon > \rho$. The proof then follows the same steps as for the ETSP ordering. For the proposed ordering, each direct path between two points must be followed by a detour path, whereas a Dubins vehicle can turn around each circle once and execute a single maneuver after covering the first circle to reach the second circle. ■

B. An Algorithm Based on Heading Discretization

We now propose an algorithm for the DTSP which is inspired from a procedure used for the curvature-constrained shortest path problem, see [5]. It chooses *a priori* a finite set of possible headings at each point. Suppose, for simplicity, that we choose K headings for each point. We then construct a graph with n clusters corresponding to the n waypoints, and each cluster containing K nodes corresponding to the choice of headings. Then, we compute the Dubins distances between configurations corresponding to pairs of nodes in distinct clusters. Finally, we would like to compute a tour through the n clusters which contains exactly one point in each cluster. This problem is called the generalized asymmetric traveling salesman problem, and can be reduced to a standard asymmetric traveling salesman problem (ATSP) over nK nodes [26]. This ATSP can in turn be solved directly using available software such as Helsgaun's implementation of the Lin-Kernighan heuristic [27], or using the $\log n$ approximation algorithm of Frieze *et al.* [28].

C. Performance Bound for $K = 1$

Suppose that we choose $K = 1$ in the previous paragraph. Our algorithm can then be described as follows:

1) Fix the headings at all points, say to 0, or by choosing them randomly uniformly in $[-\pi, \pi)$, independently for each point.
2) Compute the $n(n-1)$ Dubins distances between all pairs of points.
3) Construct a complete graph with one node for each point and edge weights given by the Dubins distances.
4) We obtain a *directed* graph where the edges satisfy the triangle inequality. Compute an exact or approximate solution for the asymmetric TSP on this graph.

Next we derive an upper bound on the approximation ratio provided by this algorithm. Let us first introduce some results and notation that will be used in this derivation. We denote the Euclidean distance between two locations $X = (x, y)$ and $X' = (x', y')$ by $E(X, X')$. The Dubins distance between two configurations $\chi = (X, \theta)$ and $\chi' = (X', \theta')$ is denoted $D(\chi, \chi')$ (note that $D(\chi, \chi') \neq D(\chi', \chi)$). The first theorem is taken from [14, Theorem 3.4].

Theorem 4: There exists a constant $\kappa \in [2.657, 2.658]$ such that for any two configurations $\chi = (X, \theta)$ and $\chi' = (X', \theta')$ we have

$$D(\chi, \chi') \leq E(X, X') + \kappa\pi\rho.$$

From the trivial bound $D(\chi, \chi') \geq E(X, X')$, we obtain the following corollary of Theorem 4.

Corollary 5: Consider two choices of headings $\theta, \hat{\theta}$ at point X and $\theta', \hat{\theta}'$ at point X', with corresponding configurations $\chi, \hat{\chi}, \chi', \hat{\chi}'$. Then we have

$$D(\hat{\chi}, \hat{\chi}') \leq \left(1 + \frac{\kappa\pi\rho}{E(X, X')}\right) D(\chi, \chi').$$

Now denote by $\{\hat{\theta}_i\}_{i=1}^n$ the headings fixed in the first step of the algorithm, and by $\{\hat{\chi}_i = (X_i, \hat{\theta}_i)\}_{i=1}^n$ the corresponding configurations at the waypoint $\{X_i = (x_i, y_i)\}_{i=1}^n$. Let ϵ be the minimum Euclidean distance between any two waypoints

$$\epsilon = \min_{i \neq j} E(X_i, X_j).$$

As in the previous corollary, since $D(\hat{\chi}_j, \hat{\chi}_i) \geq E(X_j, X_i) = E(X_i, X_j)$, we have

$$\max_{i \neq j} \frac{D(\hat{\chi}_i, \hat{\chi}_j)}{D(\hat{\chi}_j, \hat{\chi}_i)} \leq 1 + \frac{\kappa\pi\rho}{\epsilon}. \quad (2)$$

With this bound on the arc distances, we can use a modified version of Christofides algorithm, also due to Frieze *et al.* [28], to obtain a $3/2\,(1 + \kappa\pi\rho/\epsilon)$ approximation for the ATSP in step 4. The time complexity of the first three steps of our algorithm is $O(n^2)$. To solve the ATSP, we can run the two algorithms of Frieze *et al.* [28] and choose the tour with minimum length, thus obtaining an approximation ratio of $\min(\log n, 3/2\,(1 + \kappa\pi\rho/\epsilon))$. This step solving the ATSP runs in time $O(n^3)$, so overall the running time of our algorithm is $O(n^3)$. The following theorem then describes the approximation ratio of our algorithm.

Theorem 6: Given a set of n points in the plane, the algorithm described above with $K = 1$ returns a Dubins traveling salesman tour with length within a factor

$$\min\left(\left(1 + \frac{\kappa\pi\rho}{\epsilon}\right)\log n, \frac{3}{2}\left(1 + \frac{\kappa\pi\rho}{\epsilon}\right)^2\right)$$

of the length of the optimum tour. The running time of this algorithm is $O(n^3)$.

Proof: Call OPT the optimal value of the DTSP, σ^* the corresponding optimal permutation specifying the order of the waypoints, and $\{\chi_i^*\}_{i=1}^n$ the optimal configurations. We have

$$\begin{aligned} OPT &= \sum_{i=1}^{n-1} D(\chi^*_{\sigma^*(i)}, \chi^*_{\sigma^*(i+1)}) + D(\chi^*_{\sigma^*(n)}, \chi^*_{\sigma^*(1)}) \\ &=: L(\{\chi_i^*\}_{i=1}^n, \sigma^*). \end{aligned}$$

Considering the permutation σ^* for the graph problem (where the edge weights are the distances $\{D(\hat{\chi}_i, \hat{\chi}_j)\}$) and $\hat{\sigma}^*$ the optimal permutation for the graph problem, we have

$$L(\{\hat{\chi}_i\}, \hat{\sigma}^*) \leq L(\{\hat{\chi}_i\}, \sigma^*) \leq \left(1 + \frac{\kappa\pi\rho}{\epsilon}\right) L(\{\chi_i^*\}, \sigma^*)$$

where the last inequality follows from Corollary 5. We do not obtain the optimal permutation for the ATSP on the graph in general, instead we use the approximation algorithm mentioned above. Calling $\hat{\sigma}$ the permutation obtained, we have

$$\begin{aligned} &L(\{\hat{\chi}_i\}, \hat{\sigma}) \\ &\quad \leq \min\left(\log n, \frac{3}{2}\left(1 + \frac{\kappa\pi\rho}{\epsilon}\right)\right) L(\{\hat{\chi}_i\}, \hat{\sigma}^*) \\ &\quad \leq \left[\min\left(\left(1 + \frac{\kappa\pi\rho}{\epsilon}\right)\log n, \frac{3}{2}\left(1 + \frac{\kappa\pi\rho}{\epsilon}\right)^2\right)\right] OPT. \end{aligned}$$

■

We note that the bound provided by Theorem 6 is in fact worse than the bound available for the "Alternating Algorithm" (AA) of Savla *et al.* [17]. AA uses the optimal ETSP ordering, and keeps every other edge in the optimal Euclidean tour. These edges are straight lines followed by the vehicle, which must then connect the end of a straight line with the start of the next one by a Dubins path. The theoretical performance bound for our algorithm matches that of AA if we assume that the ATSP in step 4 is solved to optimality. However, experiments suggest that our algorithm performs in fact significantly better in general than AA when the waypoint distances are smaller than the turning radius (see Section V). Moreover, our algorithm can be used to potentially improve on the performance of any algorithm, such as AA. Indeed, we can fix the headings in step 1 to be those chosen by the particular algorithm. Then, assuming that the ATSP in step 4 can be solved to optimality, we obtain a tour at least as good as the one produced by the initial algorithm.

D. On the ρ/ϵ Term in Approximation Ratios

Note that every approximation ratio mentioned so far contains a ρ/ϵ term, where ϵ is the minimum Euclidean distance between any two waypoints. This term is particularly problematic for densely distributed waypoints, although it seems in general too pessimistic for points in general positions. Now the naive algorithm which considers all possible waypoint permutations and runs for each permutation the algorithm of Lee *et al.* to determine the headings [10] provides a tour within 5.03 of the optimum and runs in time $\Omega(n \cdot n!)$. We have also obtained an algorithm that runs faster than the naive algorithm, in time $O(n^2 \cdot 2^n)$, and achieves an approximation ratio of $O(\log n)$ [29]. Designing an efficient approximation algorithm for the DTSP with an approximation ratio free from this ρ/ϵ term is an open problem.

V. Numerical Simulations

Fig. 3 presents simulation results comparing the practical performance of different algorithms proposed for the DTSP. Points are generated randomly and uniformly in a 10×10 square and we fix $\rho = 1$. All the TSP tours (symmetric and asymmetric) are computed using the software LKH [27]. We compare the performance of the Alternating Algorithm (AA) [17], the nearest neighbor heuristic (NN) of Section II,

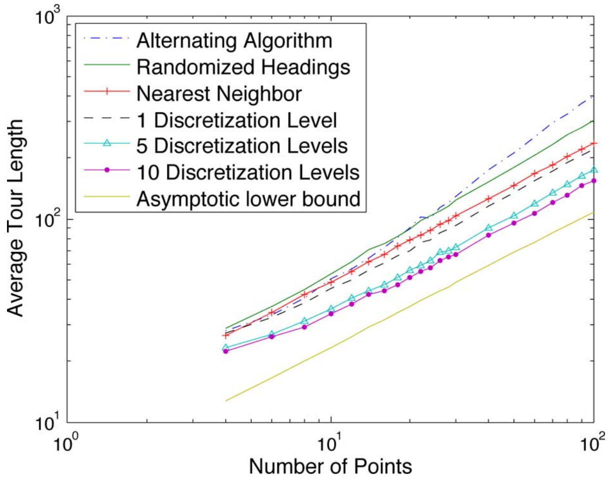
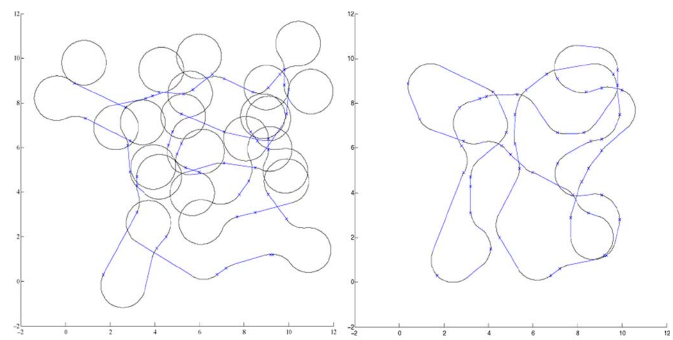


Fig. 3. Average tour length versus number of points n in a 10×10 square, on a log-log scale. The average is taken over 30 experiments for each given number of points. Note that with $n \geq 40$, the worst tour length observed among the 30 experiments was at most 15% longer than the average shown here, for all algorithms. The heading of AA is always included in the set of discrete headings in our algorithm (Section IV-C), except for the randomized case. Hence the difference between the AA curve and the one-discretization level curve shows the performance improvement obtained by only changing the waypoint ordering.

TABLE I
EXPERIMENTAL GROWTH OF THE AVERAGE DUBINS TOUR LENGTHS FOR THE DIFFERENT ALGORITHMS, WHEN THE WAYPOINTS ARE DISTRIBUTED RANDOMLY AND UNIFORMLY IN A 10-BY-10 SQUARE (EXTRACTED FROM FIG. 3)

Algorithm	Average Dubins tour length
Alternating Algorithm	$5.3 \times n^{0.94}$
Nearest Neighbor Heuristic	$9.9 \times n^{0.69}$
Randomized Headings + ATSP	$9.3 \times n^{0.75}$
AA + ATSP with 1 discretization level	$8.2 \times n^{0.71}$
5 discretization levels	$6.7 \times n^{0.7}$
10 discretization levels	$6.6 \times n^{0.68}$



(a) (b)

Fig. 4. Dubins tours through 50 points randomly distributed in a 10×10 square. The turning radius of the vehicle is 1. (a) Alternating Algorithm (b) 10 discretization levels.

and the algorithm described in Section IV-B. Compared to our algorithm with the headings chosen randomly in step 1, we see that the performance of AA is similar for low point densities. In fact, AA clearly outperforms the randomized heading version of the algorithm if the waypoints are very sparsely distributed and the optimal tour tends to become the same as the optimal Euclidean tour. However, in scenarios with waypoints densely distributed with respect to the vehicle turning radius, which arise for example for UAVs in urban environments, or loitering weapons flying at high speed [19], large performance gains can be obtained by our algorithms over AA.

As mentioned at the end of Section IV-C, even with just one discretization level, we can use the same headings as AA, and compute the solution of the ATSP using these headings (see step 4 in Section IV-C). This clearly always performs at least as well as AA, and the figure shows the significant performance improvement due to only changing the ordering, as the number of points increases. Also shown on the figure are the performance curves for an increasing number of discretization levels. With 5 discretization levels, a tour through 100 points can be computed on a standard laptop in about one minute, requiring the solution of an ATSP with 500 points, well below the limits of state-of-the-art TSP software. The asymptotic lower bound shown is taken from [15]. In particular, it is known that for the specific case of waypoints uniformly distributed in a rectangle, as in our experiments, the average length of the DTSP scales as $n^{2/3}$ [14], [15]. In fact in this case Savla *et al.* [14] have proposed an algorithm that is not based on the ETSP ordering and returns a tour whose length is within a constant factor of the optimum (in Table I, the theoretical upper bound for this algorithm with our problem parameters is $193n^{2/3}$ but numerical experiments seem to suggest a smaller constant [14]).

Table I shows the empirical growth rates for the different algorithms obtained by linear regression based on the simulation results presented in Fig. 3. With 10 discretization levels, the rate is close to optimal on such random inputs. Note also the good asymptotic performance of the nearest neighbor heuristic, which moreover is much easier to compute than the other heuristics. It is useful in practice to include the headings of this nearest neighbor heuristic as part of the set of headings used in the discretization of Section IV-B. Finally, Fig. 4 shows an example of tours through 50 points computed using the Alternating Algorithm and our algorithm using 10 discretization levels.

VI. CONCLUSION

This technical note provides a proof of the NP-hardness of the Dubins Traveling Salesman Problem, thereby justifying the focus on approximation algorithms. We describe hard problem instances for which the existing algorithms approximate the best Dubins tour only within a factor $\Omega(n)$. We also discuss how one can essentially improve the practical performance of the existing heuristics by solving an asymmetric traveling salesman problem once a set of possible headings at each point is chosen. Establishing the existence of a polynomial-time algorithm that provides an approximation ratio free from the ρ/ϵ factor appearing in this technical note remains an open question.

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Comments on "Structural Invariant Subspaces of Singular Hamiltonian Systems and Nonrecursive Solutions of Finite-Horizon Optimal Control Problems"

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***Abstract*—In this note it is shown that the main result of [1], concerning a characterization of a pair of structural invariant subspaces associated with the extended symplectic system, is a particular case of a result presented in [2] and [3] within a more general and rigorous context. We also analyse the proof of the main result of [1], and the way such result is used to accomodate the boundary conditions in the solution of a finite-horizon linear quadratic optimal control problem.**

***Index Terms*—Algebraic Riccati equations (AREs), extended symplectic difference equation (ESDE), Hamiltonian differential equations (HDE), linear-quadratic (LQ).**

I. Introduction

The paper under discussion presents a method to solve a finite-horizon general linear-quadratic (LQ) optimal control problem by using a formula which parameterizes the set of trajectories generated by the extended symplectic difference equation (ESDE). The idea of solving finite-horizon LQ problems by exploiting expressions of the trajectories generated by the Hamiltonian differential equations (HDE) in the continuous time or the ESDE in the discrete time is not new. It originated in [5] and [6] for the continuous time, and in [7] for the discrete time. We must note that these papers are not cited in [1]. The expressions parameterizing the trajectories of HDE and ESDE given in these first contributions hinge on particular solutions of the associated algebraic Riccati equations (AREs). While controllability of the given system was required in these first papers, because both the stabilizing and anti-stabilizing solutions of the ARE were involved, in more recent publications it was shown that generalizations of the same technique are possible under much milder assumptions: namely, sign-controllability in the continuous case, see [2, Section 1.4] and [4], and modulus controllability in the discrete case, see [2, Section 2.4] and [3]. These assumptions are to date the weakest conditions that guarantee existence of solutions of an ARE. Sign-controllability and modulus-controllability are weaker assumptions than controllability, stabilizability and anti-stabilizability. They generically hold even in the extreme case when B is the zero matrix. This active stream of research not only produced the theoretical background which is necessary for the application of these techniques to more general types of systems, but it also considerably enlarged the range of optimization problems that can be successfully addressed. In particular, in [4] and [3] it is shown that the parameterization technique described above can be applied to (continuous and discrete) finite-horizon LQ problems with the most general form of affine constraints at the end-points (thus encompassing the standard, the fixed end-point and the point-to-point

Manuscript received March 21, 2010; revised September 22, 2010, January 29, 2011, and April 27, 2011; accepted July 10, 2011. Date of publication July 25, 2011; date of current version December 29, 2011. Recommended by Associate Editor L. Marconi.

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Digital Object Identifier 10.1109/TAC.2011.2162894