

On the Curvature-Constrained Traveling Salesman Problem

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Abstract

We study the traveling salesman problem for a Dubins car. We prove that this problem is NP-hard, and provide lower bounds on the approximation ratio achievable by some recently proposed heuristics. In particular, the approximation ratio achievable by any algorithm that always follows the order optimal for the Euclidean metric is $\Omega(n)$. We also describe new algorithms for this problem based on heading discretization, and evaluate their performance numerically.

I. INTRODUCTION

In an instance of the traveling salesman problem (TSP) we are given the distances d_{ij} between any pair of n points. The problem is to find the shortest tour visiting every point exactly once. We also call this problem the tour-TSP to distinguish it from the path-TSP, where the requirement that the vehicle must start and end at the same point is removed. This famously intractable problem is often encountered in robotics and typically solved by the higher decision-making levels in the common layered controller architectures. The dynamics of the robot are usually not taken into account at this stage and the mission planner might typically chose to solve the TSP for the Euclidean metric (ETSP), i.e., the distances d_{ij} represent the Euclidean distances

This work was supported by Air Force - DARPA - MURI award 009628-001-03-132 and Navy ONR award N00014-03-1-0171. Preliminary versions of this work appeared in [1], [2].

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between waypoints. This can result in poor overall performance, since the sequence of points chosen by the algorithm can be very hard to follow for the physical system.

In order to improve the performance of unmanned aerial systems in particular, researchers are working on integrating the high level problem of mission planning and the low level problem of path planning [3]. We obtain a motion planning problem with differential constraints, for which most of the existing work concerns the shortest path problem, see, e.g., [4]. In this note we consider the traveling salesman problem for the Dubins vehicle (DTSP). The Dubins vehicle can only move forward in the plane, at constant speed, and has a limited turning radius. This is a good kinematic model for fixed wing aircraft, which can produce a waypoint sequence that is relevant for the low-level controller of these vehicles. At the same time, we have for example an analytical characterization of the shortest path between any two configurations of the Dubins vehicle, giving hope that efficient algorithms with good performance can be designed to solve the TSP.

A stochastic version of the DTSP for which the points are distributed randomly and uniformly in the plane was considered in [5], [6], [7]. Here however, we focus on algorithms and worst-case bounds for the more standard problem where no input distribution is given. In that case, most of the recently proposed algorithms seem to build on a preliminary solution obtained for the ETSP [8], [9], [10], [11]. We refer the reader to our more detailed survey of the existing algorithms in [2].

Contributions of this work. In this note, we first prove that the DTSP is NP-hard, thus justifying the work on heuristics and algorithms that approximate the optimal solution. On the negative side, we give some lower bounds on the approximation ratio achievable by recently proposed heuristics. Following a tour based on the ETSP ordering or the ordering of Tang and Özgüner [3] cannot achieve an approximation ratio better than $\Omega(n)$ ¹. The same is true for the nearest neighbor heuristic, in contrast to the ETSP where it achieves a $O(\log n)$ approximation [12]. Then we propose an algorithm based on heading discretization, which emerges naturally from the previous work on the curvature-constrained shortest path. Its theoretical performance is of the same type as for these shortest-path algorithms and does not improve on the previously

¹We say $f(n) = O(g(n))$ if there exists $c > 0$ such that $f(n) \leq cg(n)$ for all n , and $f(n) = \Omega(g(n))$ if there exists $c > 0$ such that $f(n) \geq cg(n)$ for all n .

mentioned heuristics. However numerical simulations show a significant performance improvement in randomly generated instances over other heuristics when the inter waypoint distances are smaller than the turning radius of the vehicle.

The rest of this paper is organized as follows. We recall some facts about the Dubins paths in section II and reduce the DTSP to a finite dimensional optimization problem. This section also contains the lower bound on approximation ratios for various heuristics. In Section III we show that the DTSP is NP-hard. Section IV describes our algorithms based on heading discretization. The basic version returns in time $O(n^3)$ a tour within $O\left(\min\left(\left(1 + \frac{\rho}{\varepsilon}\right) \log n, \left(1 + \frac{\rho}{\varepsilon}\right)^2\right)\right)$ of the optimum, where ρ is the minimum turning radius of the vehicle and ε is the minimum Euclidean distance between any two waypoints. Finally, section V discusses some numerical simulations.

II. HEURISTICS PERFORMANCE

A Dubins vehicle in the plane has its configuration described by its position and heading $(x, y, \theta) \in \mathbb{R}^2 \times (-\pi, \pi]$. Its equations of motion are

$$\dot{x} = v_0 \cos(\theta), \quad \dot{y} = v_0 \sin(\theta), \quad \dot{\theta} = \frac{v_0}{\rho} u, \quad \text{with } u \in [-1, 1].$$

Without loss of generality, we can assume that the speed v_0 of the vehicle is normalized to 1. ρ is the minimum turning radius of the vehicle. u is the available control. The DTSP asks, for a given set of points in the plane, to find the shortest tour through these points that is feasible for a Dubins vehicle. Since we show below that this problem is NP-hard, we will focus on the design of approximation algorithms. An α -approximation algorithm (with $\alpha \geq 1$) for a minimization problem is an algorithm that produces *in polynomial time*, on any instance of the problem with optimum OPT , a feasible solution whose value Z is within a factor α of the optimum, i.e., such that $OPT \leq Z \leq \alpha OPT$.

Dubins [13] characterized curvature constrained shortest paths between an initial and a final configuration. Let P be a feasible path. We call a nonempty subpath of P a C -segment or an S -segment if it is a circular arc of radius ρ or a straight line segment, respectively. We paraphrase the following result from Dubins:

Theorem 1 ([13]): An optimal path between any two configurations is of type CCC or CSC, or a subpath of a path of either of these two types. Moreover, to be optimal, a CCC path must have its middle arc of length greater than $\pi\rho$.

In the following, we will refer to these minimal-length paths as Dubins paths. When a subpath is a C -segment, it can be a left or a right hand turn: denote these two types of C -segments by L and R respectively. Then we see from Theorem 1 that to find the minimum length path between an initial and a final configuration, it is enough to find the minimum length path among six paths, namely among $\{LSL, RSR, RSL, LSR, RLR, LRL\}$. Each of these paths can be explicitly computed and therefore finding the optimum path and length between any two configurations can be done in constant time.

Given this result, solving the DTSP is reduced to choosing a permutation of the points specifying in which order to visit them, as well as choosing a heading for the vehicle at each of these points. If the Euclidean distance between the waypoints to visit is large with respect to ρ , the choice of headings has a limited impact and the DTSP and ETSP behave similarly. This has motivated work using for the DTSP the waypoint ordering optimal for the ETSP [8], [9], [10], and concentrating on the choice of headings. Theorem 2 provides a limit on the performance one can achieve using such a technique, which becomes particularly significant in the case where the points are densely distributed with respect to ρ .

Before stating the theorem, we describe another simple heuristic, called the greedy or nearest neighbor heuristic. We start with an arbitrary point, and choose its heading arbitrarily, fixing an initial configuration. At each step, we find the point not yet on the path which is closest for the Dubins metric to the last added configuration. This is possible since we also have a complete characterization of the Dubins distance between an initial configuration and a final point with free heading [14]. We add this closest point with the associated optimal arrival heading. When all nodes have been added to the path, we add a Dubins path connecting the last configuration and the initial configuration. It is known that for the ETSP, which is a particular case of the symmetric TSP, the nearest neighbor heuristic achieves a $O(\log n)$ approximation ratio [12].

Theorem 2: Any algorithm for the DTSP which always follows the ordering of points that is optimal for the ETSP has an approximation ratio R_n which is $\Omega(n)$. If we impose a lower bound ε sufficiently small on the minimum Euclidean distance between any two waypoints, then there exist constants C, C' , such that the approximation ratio is not better than $\frac{C}{\varepsilon + (C'/n)}$. These statements are also true for the nearest neighbor heuristic.

Proof: Consider the configuration of points shown in fig. 1(a). Let n be the number of

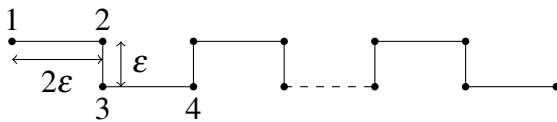
points, and suppose $n = 4m$, m an integer. For clarity we focus on the path-TSP problem but extension to the tour-TSP case is easy, by adding a similar path in the reverse direction. The optimal Euclidean path-TSP is shown on the figure as well. Suppose now that a Dubins vehicle tries to follow the points in this order, and suppose ε is sufficiently small. Then for each sequence of 4 consecutive points, two on the upper line and two on the lower line, the Dubins vehicle will have to execute a maneuver of length at least C , where C is a constant of order $2\pi\rho$. For instance, if the vehicle tries to go through the two top points without a large maneuver, it will exit the second point with a heading almost horizontal and will have to make a large turn to catch the third point on the lower line. This discussion can easily be formalized using the results on the final headings of direct Dubins paths from [15]. Hence the length of the Dubins path following the Euclidean TSP ordering will be greater than mC .

On the other hand, a Dubins vehicle can simply go through all the points on the top line, execute a U-turn of length C' of order $2\pi\rho$, and then go through the points on the lower line, providing an upper bound of $2n\varepsilon + C'$ for the optimal solution. So we deduce that the worst case approximation ratio of the algorithm is at least:

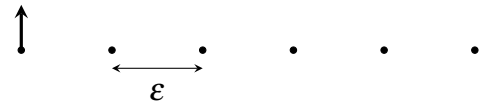
$$R_n \geq \frac{\frac{n}{4}C}{2n\varepsilon + C'}.$$

But we can choose ε as small as we want, in particular we can choose $\varepsilon \leq 1/n$ for problem instances with n points, and thus $R_n = \Omega(n)$.

For the nearest-neighbor heuristic, we use the configuration shown on fig. 1(b), for ε small enough. The figure shows the first point and heading chosen by the algorithm. The proof is similar to the argument above and left to the reader. It essentially uses the fact that points inside the circles of radius ρ tangent to the initial headings of a Dubins path starting at a point P are at Dubins distance greater than $\pi\rho$ from P [14]. ■



(a) Waypoint configuration and associated path-ETSP used in the proof of theorem 2.



(b) Configuration to analyze the performance of the nearest-neighbor heuristic.

Fig. 1. Waypoint configurations used as counter-examples.

Remark 3: In [3], Tang and Özgüner do not use the ETSP ordering, but instead, construct the geometric center of the waypoints, calculate the orientations of the waypoints with respect to that center, and traverse the points in order of increasing orientations. It is easy to adapt Theorem 2 to this case, closing the path of fig. 1(a) into a cycle for example.

Remark 4: Using the notation of the theorem, the “Alternating Algorithm” proposed in [8] achieves an approximation ratio of approximately $1 + 3.48\frac{\rho}{\epsilon}$ [6].

Remark 5: If the waypoint ordering is decided a priori, for example using the solution for the ETSP, [15] describes a linear time algorithm to choose the headings which provides a tour with length at most 5.03 times the length of the tour achievable by a choice of headings optimal for this waypoint ordering.

III. COMPLEXITY OF THE DTSP

It is usually accepted that the DTSP is NP-hard and the goal of this section is to prove this claim rigorously. Note that adding the curvature constraint to the Euclidean TSP could well make the problem easier². Hence, the statement does not follow trivially from the NP-hardness of the ETSP [17], [18]. In the proof of theorem 6, we consider, without loss of generality, the decision version of the problem, which we also call DTSP. That is, given a set of points in the plane and a number $L > 0$, DTSP asks if there exists a tour for the Dubins vehicle visiting all these points exactly once, of length at most L .

Theorem 6: Tour-DTSP and path-DTSP are NP-hard.

Proof: This is a corollary of Papadimitriou’s proof of the NP-hardness of ETSP, to which we refer [17]. First recall the Exact Cover Problem: given a family F of subsets of the finite set U , is there a subfamily F' of F , consisting of disjoint sets, such that F' covers U ? This problem is known to be NP-complete [19]. Papadimitriou described a polynomial-time reduction of Exact Cover to ETSP. That is, given an instance of the Exact Cover problem, we can construct an instance of the Euclidean Traveling Salesman Problem and a number L such that the Exact Cover problem has a solution if and only if the ETSP has an optimal tour of length less than or equal to L . The important fact to observe however, is that if Exact Cover does not have a solution, Papadimitriou’s construction gives an instance of the ETSP which has an optimal tour

²as in the bitonic TSP [16, p. 364] for example.

of length $\geq (L + \delta)$, for some $\delta > 0$, and not just $> L$. More precisely, letting $a = 20$ exactly as in his proof, we can take $0 < \delta < \sqrt{a^2 + 1} - a$.

Now from [8], there is a constant C such that for any instance $\mathcal{P}$ of ETSP with n points and length $ETSP(\mathcal{P})$, the optimal DTSP tour for this instance has length less than or equal to $ETSP(\mathcal{P}) + Cn$. Then if we have n points in the instance of the ETSP constructed as in Papadimitriou's proof, we simply rescale all the distances by a factor $2Cn/\delta$. If Exact Cover has a solution, the ETSP instance has an optimal tour of length no more than $2CnL/\delta$ and so the curvature constrained tour has a length of no more than $2CnL/\delta + Cn$. If Exact Cover does not have a solution, the ETSP instance has an optimal tour of length at least $2CnL/\delta + 2Cn$, and the curvature constrained tour as well. So Papadimitriou's construction, rescaled by $2Cn/\delta$ and using $2CnL/\delta + Cn$ instead of L , where n is the number of points used in the construction, provides a reduction from Exact Cover to DTSP. ■

IV. AN ALGORITHM BASED ON HEADING DISCRETIZATION

We now propose an algorithm for the DTSP which is inspired from a procedure used for the curvature-constrained shortest path problem [20]. It chooses a priori a finite set of possible headings at each point. Suppose, for simplicity, that we choose K headings for each point. We construct a graph with n clusters corresponding to the n waypoints, and each cluster containing K nodes corresponding to the choice of headings. Then, we compute the Dubins distances between configurations corresponding to pairs of nodes in distinct clusters. Finally, we would like to compute a tour through the n clusters which contains exactly one point in each cluster. This problem is called the generalized asymmetric traveling salesman problem, and can be reduced to a standard asymmetric traveling salesman problem (ATSP) over nK nodes [21]. This ATSP can in turn be solved directly using available software such as Helsgaun's implementation of the Lin-Kernighan heuristic [22], or using the $\log n$ approximation algorithm of Frieze et al. [23].

We found this method to perform very well in practice. The type of approximation result one can expect with such an a priori discretization of the headings is the same as in the shortest path case, that is, the path obtained is close to the optimum if the optimum is "robust". This means that the discretization must be sufficiently fine so that the algorithm can find a Dubins path close to the optimum which is of the same type, as defined in section II (see [24]). A limitation of the algorithm is that we must solve a TSP over nK points instead of n .

A. Performance Bound for $K = 1$.

Suppose that we choose $K = 1$ in the previous paragraph. Our algorithm can then be described as follows:

- 1) Fix the headings at all points, say to 0, or by choosing them randomly uniformly in $[-\pi, \pi)$, independently for each point.
- 2) Compute the $n(n-1)$ Dubins distances between all pairs of points.
- 3) Construct a complete graph with one node for each point and edge weights given by the Dubins distances.
- 4) We obtain a *directed* graph where the edges satisfy the triangle inequality. Compute an exact or approximate solution for the asymmetric TSP on this graph.

Next we derive an upper bound on the approximation ratio provided by this algorithm. If we look for a polynomial time algorithm, we must be satisfied with an approximate solution to the ATSP in step 4. There is however some additional information that we can exploit once the headings have been fixed, which differentiates the problem from a general ATSP. We have the following:

Lemma 7: Let $\hat{d}_{ij}$ denote the Dubins distance from point $X_i = (x_i, y_i)$ to point $X_j = (x_j, y_j)$ in the plane after the headings $\{\theta_k\}_{k=1}^n$ have been fixed. Then:

$$\max_{i,j} \frac{\hat{d}_{ij}}{\hat{d}_{ji}} \leq 1 + \frac{4\pi\rho}{\varepsilon},$$

where ε is, for the rest of this section, the minimum Euclidean distance between any two waypoints.

Proof: This result follows from the easy inequality $\hat{d}_{ij} \geq \varepsilon$ and from the following fact:

$$\hat{d}_{ij} \leq \hat{d}_{ji} + 4\pi\rho.$$

To show this, consider the Dubins path from the configuration (X_i, θ_i) to (X_j, θ_j) . We construct a feasible path (not necessarily optimal) for the Dubins vehicle, to return from (X_j, θ_j) to (X_i, θ_i) . On the forward path, the Dubins vehicle moves along an initial circle (of radius ρ), a straight line or a middle circle, and a final circle. For a CCC path, it is easy to find a return path on the same 3 circles, using the small arc on the middle circle (such a return path is not optimal). For the CSC paths, the straight line is tangent to the following 4 circles: two circles are used as turning circles in the forward path, and the other two circles are symmetric with respect to the

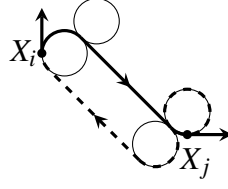


Fig. 2. A feasible return path for a RSL Dubins path.

line. We consider return paths for the different types of Dubins paths, which use these 4 circles and either the straight line of the forward path or a parallel segment of the same length. The bound is shown to hold in all cases. An example for an RSL path is provided on Fig. 2. ■

With this bound on the arc distances, we can use a modified version of Christofides algorithm, also due to Frieze et al. [23], to obtain a $\frac{3}{2} \left(1 + \frac{4\pi\rho}{\varepsilon}\right)$ approximation for the ATSP in step 4. The time complexity of the first three steps of our algorithm is $O(n^2)$. To solve the ATSP, we can run the two algorithms of Frieze et al. and choose the tour with minimum length, thus obtaining an approximation ratio of $\min\left(\log n, \frac{3}{2} \left(1 + \frac{4\pi\rho}{\varepsilon}\right)\right)$. The algorithm for solving the ATSP runs in $O(n^3)$, so overall the running time of our algorithm is $O(n^3)$. To analyze the approximation performance of this algorithm, we will use the following bound.

Lemma 8: Let d_{ij} be the length of the Dubins path between two configurations (X_i, θ_i) and (X_j, θ_j) and let ε_{ij} be the Euclidean distance between X_i and X_j . Let $\hat{d}_{ij}$ be the Dubins distance between the two points after fixing the headings in the first step of the algorithm. If the headings are chosen deterministically, we have

$$\hat{d}_{ij} \leq \left(1 + 2\pi\rho \max\left\{\frac{4}{\varepsilon_{ij}}, \frac{7}{3\max(\varepsilon_{ij}, \pi\rho)}\right\}\right) d_{ij} =: C_1 d_{ij}, \quad (1)$$

whereas if they are chosen randomly and uniformly, we obtain

$$E[\hat{d}_{ij}] \leq \left(1 + \frac{13.58\rho}{\varepsilon_{ij}}\right) d_{ij} =: C_2 d_{ij}. \quad (2)$$

Proof: The bound follows from the discussion in section 4.8 of [20]³, which we briefly summarize. The headings at X_i and X_j , as fixed by the algorithm, can differ from the given

³It appears that eq. (4.38) and (4.48) in [20] contain a typographical error, namely a ρ is missing.

heading θ_i and θ_j by $\delta_i, \delta_j \in [-\pi, \pi)$. This translates into a path difference

$$|d_{ij} - \hat{d}_{ij}| \leq \rho \left[\max \{A_i, B_i\} + \max \{A_j, B_j\} \right],$$

where

$$A_i = 3|\delta_i| + \pi \left| \sin\left(\frac{\delta_i}{2}\right) \right|, B_i = |\delta_i| + 4 \arccos \left(1 - \frac{|\sin(\delta/2)|}{2} \right),$$

and we substitute δ_i to δ_j to get A_j, B_j . The A and B terms come from the analysis of perturbations of CSC paths with opposite initial and final turning directions and CCC paths respectively. Perturbations of a CSC path with identical initial and final turning directions are dominated by the A terms. To obtain (1), we take the worst possible values $\delta_i = \delta_j = -\pi$. Moreover, for the second part of the max term, the fact that a CCC optimal path must have minimum length $\pi\rho$ was used. The bound (2) follows from the fact that in the randomized heading assignment, the perturbations δ_i, δ_j are uniformly distributed in $[-\pi, \pi)$, and we can then compute

$$E \left[\max \{A_i, B_i\} + \max \{A_j, B_j\} \right] = 13.58.$$

■

The following theorem then describes the approximation ratio of our algorithm.

Theorem 9: Given a set of n points in the plane, the algorithm described above with $K = 1$ returns a Dubins traveling salesman tour with length within a factor $O \left(\min \left(\left(1 + \frac{\rho}{\varepsilon}\right) \log n, \left(1 + \frac{\rho}{\varepsilon}\right)^2 \right) \right)$ of the length of the optimum tour. More precisely, the deterministic choice of headings guarantees an approximation ratio of

$$\left(1 + 2\pi\rho \max \left\{ \frac{4}{\varepsilon_{ij}}, \frac{7}{3 \max(\varepsilon_{ij}, \pi\rho)} \right\} \right) \times \min \left(\log n, \frac{3}{2} \left(1 + \frac{4\pi\rho}{\varepsilon} \right) \right),$$

whereas the randomized choice of headings provides an approximation ratio of

$$\left(1 + \frac{13.58\rho}{\varepsilon} \right) \min \left(\log n, \frac{3}{2} \left(1 + \frac{4\pi\rho}{\varepsilon} \right) \right)$$

in expectation. The running time of this algorithm is $O(n^3)$.

Proof: Call OPT the optimal value of the DTSP and σ^* the corresponding optimal permutation specifying the order of the waypoints. We have $OPT = \sum_{i=1}^{n-1} d_{\sigma^*(i)\sigma^*(i+1)} + d_{\sigma^*(n)\sigma^*(1)} := L(\{d_{ij}\}, \sigma^*)$. Considering the permutation σ^* for the graph problem (where the edge weights are the distances $\{\hat{d}_{ij}\}$) and $\hat{\sigma}^*$ the optimal permutation for the graph problem, we have

$$L(\{\hat{d}_{ij}\}, \hat{\sigma}^*) \leq L(\{\hat{d}_{ij}\}, \sigma^*) \leq C L(\{d_{ij}\}, \sigma^*),$$

with C equal to C_1 or C_2 as defined in lemma 8. We do not obtain the optimal permutation for the ATSP on the graph in general, instead we use the approximation algorithm mentioned above. Calling $\hat{\sigma}$ the permutation obtained, we have:

$$\begin{aligned} L(\{\hat{d}_{ij}\}, \hat{\sigma}) &\leq \min \left(\log n, \frac{3}{2} \left(1 + \frac{4\pi\rho}{\varepsilon} \right) \right) L(\{\hat{d}_{ij}\}, \hat{\sigma}^*) \\ &\leq \left(C \min \left(\log n, \frac{3}{2} \left(1 + \frac{4\pi\rho}{\varepsilon} \right) \right) \right) OPT. \end{aligned}$$

■

Remark 10: For better performance in practice, one should try to run the randomized algorithm several times and choose the shortest tour obtained.

We note that the bound provided is in fact worse than the bound available for the “Alternating Algorithm” (AA), see remark 4. However, this bound does not seem to be tight, and moreover, experiments suggest that the randomized heading assignment in fact performs significantly better in general than the AA when the waypoint distances are smaller than the turning radius (see section V).

B. On the ρ/ε Term in Approximation Ratios

Note that every approximation ratio mentioned so far contains a ρ/ε term, where ε is the minimum Euclidean distance between any two waypoints. This term is particularly problematic for densely distributed waypoints, although it seems in general too pessimistic for points in general positions. Now the naive algorithm which considers all possible waypoint permutations and runs for each permutation the algorithm of Lee et al. to determine the headings [15] provides a tour within 5.03 of the optimum and runs in time $\Omega(n \cdot n!)$. We have also obtained an algorithm that runs faster than the naive algorithm, in time $O(n^2 \cdot 2^n)$, and achieves an approximation ratio of $O(\log n)$ [25]. It would be interesting to understand if one can design a polynomial-time algorithm that provides an approximation ratio free from the ρ/ε factor.

V. NUMERICAL SIMULATIONS

Fig. 3 presents simulation results comparing the practical performance of different algorithms proposed for the DTSP. Points are generated randomly in a 10×10 square. All the TSP tours

(symmetric and asymmetric) are computed using the software LKH [22]. We compare the performance of the Alternating Algorithm (AA) [8], the nearest neighbor heuristic (NN) of section II, and the algorithm described in section IV. AA is based on the optimal Euclidean ordering. It computes the optimal ETSP, and keeps every other edge in the loop. These edges are straight lines followed by the vehicle, which must then connect the end of a straight path with the start of the next one by a Dubins path. Compared to our randomized heading algorithm, we see that the performance of AA is similar for low point densities. In fact, AA clearly outperforms the randomized heading algorithm if the points are very sparsely distributed and the optimal tour tends to become the same as the optimal Euclidean tour. However, in scenarios with waypoints densely distributed with respect to the vehicle turning radius, which arise for example for UAVs in urban environments, or loitering weapons flying at high speed [11], large performance gains can be obtained by our algorithms over the AA.

Moreover for the choice of heading, even with just one discretization level, we can use the same heading as AA, and compute the solution of the ATSP (as in step 4 of paragraph IV-A) using this heading. This clearly always performs at least as well as AA, and the figure shows the significant performance improvement due to only changing the ordering, as the number of points increases. Also shown on the figure are the performance curves for an increasing number of discretization levels. With 5 discretization levels, a tour through 100 points can be computed on a standard laptop in about one minute, requiring the solution of an ATSP with 500 points, well below the limits of state-of-the-art TSP software. The asymptotic lower bound shown is taken from [7]. In particular, it is known that for waypoints uniformly distributed in a rectangle, the average length of the DTSP scales as $n^{2/3}$ [7], [6]. Empirically, we observed the growth rates shown in table I for the different algorithms. With 10 discretization levels, the rate is close to optimal on such random inputs. Note also the good asymptotic performance of the nearest neighbor heuristic, which moreover is much easier to compute than the other heuristics. Although not used for the comparisons on Fig. 3, it is worth in practice including the headings of NN as part of the set of headings used in the discretization of section IV. Finally, fig. 4 shows an example of tours through 50 points computed using the Alternating Algorithm and our algorithm using 10 discretization levels.

Algorithm	Dubins tour average length
Alternating Algorithm	$5.3 \times n^{0.94}$
Nearest Neighbor Heuristic	$9.9 \times n^{0.69}$
Randomized Headings	$9.3 \times n^{0.75}$
AA + ATSP with 1 discretization level	$8.2 \times n^{0.71}$
5 discretization levels	$6.7 \times n^{0.7}$
10 discretization levels	$6.6 \times n^{0.68}$

TABLE I

EXPERIMENTAL GROWTH OF THE AVERAGE DUBINS TOUR LENGTHS FOR THE DIFFERENT ALGORITHMS, WHEN THE WAYPOINTS ARE DISTRIBUTED RANDOMLY AND UNIFORMLY IN A 10-BY-10 SQUARE.

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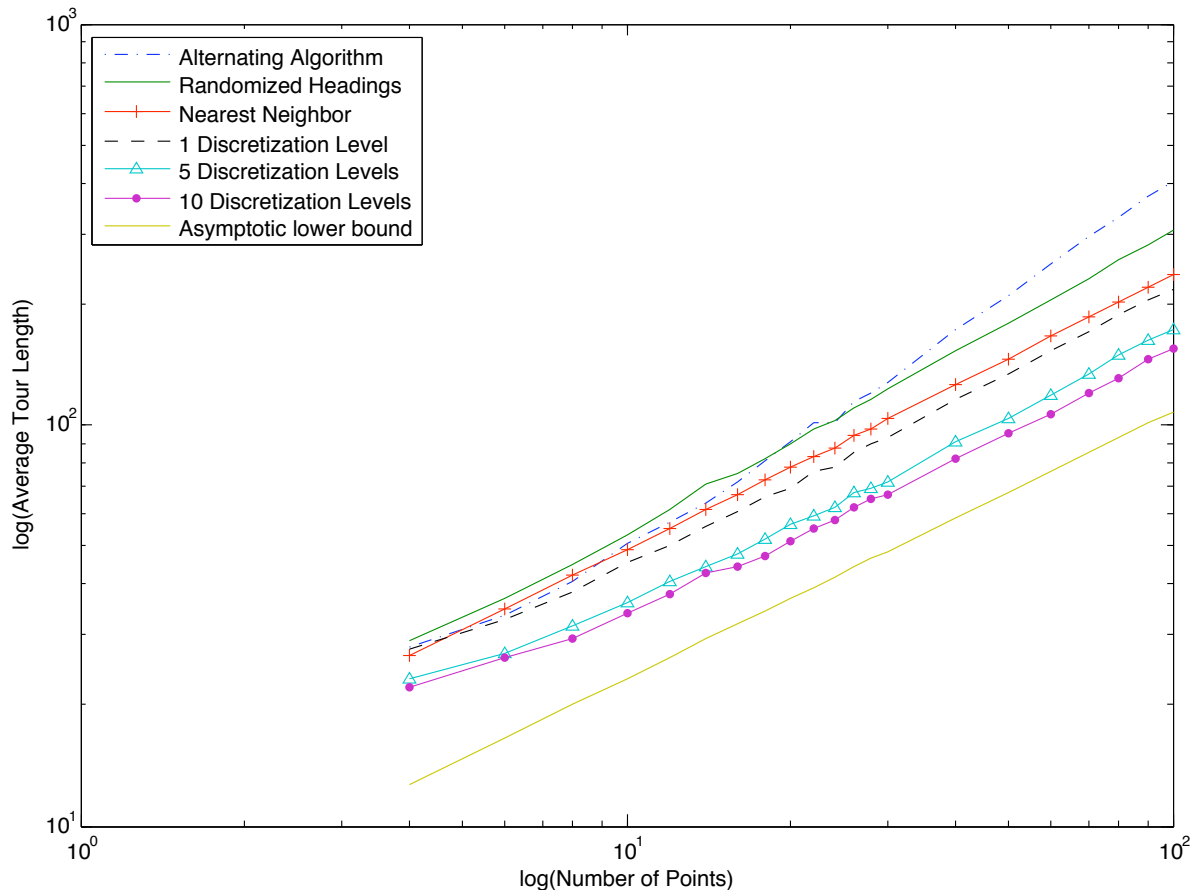
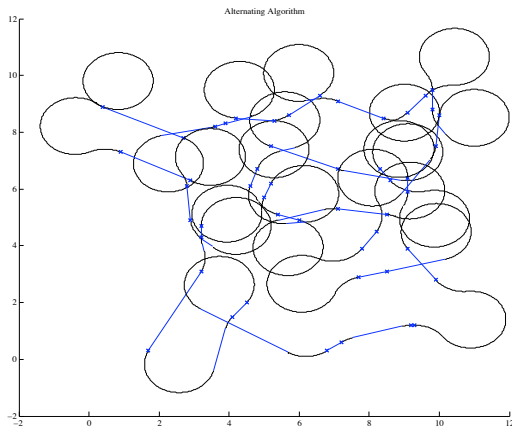
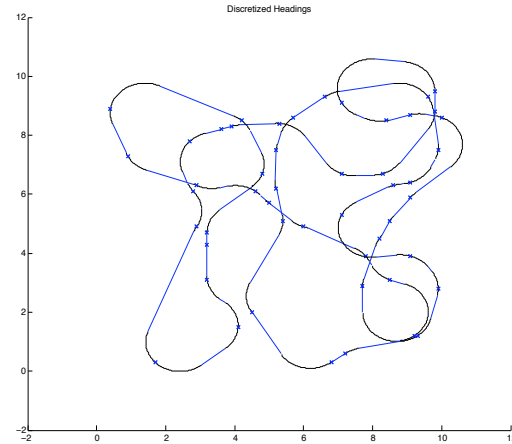


Fig. 3. Average tour length vs. number of points in a 10×10 square, on a log-log scale. The average is taken over 30 experiments for each given number of points. Except for the randomized headings, the angle of the AA is always included in the set of discrete headings. Hence the difference between the AA curve and the 1 discretization level curve shows the performance improvement obtained by only changing the waypoint ordering.

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(a) Alternating Algorithm



(b) 10 discretization levels.

Fig. 4. Dubins tours through 50 points randomly distributed in a 10×10 square. The turning radius of the vehicle is 1.

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