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A Dynamic Collective Choice Model With An Advertiser

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Abstract This paper studies a dynamic collective choice model in the presence of an advertiser, where a large number of consumers are choosing between two alternatives. Their choices are influenced by the group's aggregate choice and an advertising effect. The latter is produced by an advertiser making investments to convince as many consumers as possible to choose a specific alternative. In schools for example, teenagers' decisions to smoke are considerably affected by their peers' decisions, as well as the ministry of health campaigns against smoking. We model the problem as a Stackelberg dynamic game, where the advertiser makes its investment decision first, and then the consumers choose one of the alternatives. On the methodological side, we use the theory of mean field games to solve the game for a continuum of consumers. This allows us to describe the consumers' individual and aggregate behaviors, and the advertiser's optimal investment strategies. When the consumers have sufficiently diverse a priori opinions towards the alternatives, we show that a unique Nash equilibrium exists between them, which predicts the distribution of choices over the alternatives, and the advertiser can always make optimal investments. For a certain uniform distribution of a priori opinions, we give

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an explicit form of the advertiser's optimal investment strategy and of the consumers' optimal choices.

Keywords Mean Field Games · Stackelberg competition · Advertising

1 Introduction

Advertising is the activity or effort to promote a product or an idea. It addresses a mass of consumers making a choice between multiple products. In the past forty years, there has been much interest in developing mathematical advertising models [8,9,19,20]. The main goal is to understand the influence of the advertising process on the sales level, and develop optimal, profit-maximizing advertising strategies. However, these models investigate only the macroscopic aspect of the advertising problem, that is, the market's response (sales level) to the advertising strategies. In contrast, this paper is concerned with both the microscopic and macroscopic levels. Indeed, it tries to model the response of each consumer (microscopic level) to the advertising process, and to anticipate the market's response (macroscopic level). Our approach allows us to model, in addition to the advertising effect, some other factors that influence the individual and market's responses. These factors include the consumers' a priori opinions towards the products, and a social effect. For example, in its intent to reduce the percentage of smokers among teenagers, the government makes some investments in the form of campaigns against smoking to encourage the teenagers not to smoke. But, at the beginning of the campaign, the teenagers have different tendencies towards smoking. Moreover, a teenager's decision to smoke is considerably affected by his or her peers' decisions [22].

We model the dynamic collective choice problem with an advertiser as a Stackelberg dynamic game. It includes a large number of consumers choosing between two alternatives, while being influenced by their aggregate behavior and an advertising effect. The latter is produced by an advertiser making investments to convince as many consumers as possible to choose a specific alternative. The advertiser is "Stackelbergian" [2], that is, it makes its decision first, with the consumers deciding afterwards, i.e., advertisement precedes consumption. We use the Mean Field Games (MFG) methodology to solve the game. This allows us to describe the consumers' individual and aggregate behaviors, and the advertiser's optimal investment strategies. Moreover, when the consumers' a priori opinions towards the alternatives are sufficiently diverse, in a sense made precise in Section 3.1, our model predicts the distribution of choices over the alternatives as a unique Nash equilibrium between the consumers.

The MFG methodology is concerned with a class of dynamic games involving a large number of agents interacting through the mass effect of the group. It assumes an infinite population to which one can ascribe a deterministic although initially unknown macroscopic behavior, i.e., a given flow of population probability distributions also known as the mean field over the control horizon of interest. In view of the vanishing influence of isolated individuals, a generic agent's best response is then described by a Hamilton-Jacobi-Bellman (HJB) equation propagating backwards and parameterized by the macroscopic behavior. In turn, the macroscopic behavior satisfies a forward Fokker-Planck (FP) equation parameterized by the generic agent's best response. Candidate sustainable macroscopic behaviors are then computed as the fixed points, if they exist, of a suitable map between macroscopic behaviors. The corresponding best responses, when applied to the more realistic finite population situation, constitute, under appropriate conditions, an approximate Nash equilibrium, or $\epsilon-$Nash equilibrium [13, 14].

Definition 1 Consider N agents, a set of strategy profiles $S = S_1 \times \cdots \times S_N$ and, for each agent k, a cost function $J_k(u_1, \ldots, u_N)$, $\forall (u_1, \ldots, u_N) \in S$. A strategy profile $(u_1^*, \ldots, u_N^*) \in S$ is called an $\epsilon-$Nash equilibrium with respect to the costs J_k if there exists an $\epsilon > 0$ such that for any fixed $1 \leq i \leq N$, for all $u_i \in S_i$, we have $J_i(u_i, u_{-i}^*) \geq J_i(u_i^*, u_{-i}^*) - \epsilon$, where $u_{-i}^* = (u_1^*, \ldots, u_{i-1}^*, u_{i+1}^*, \ldots, u_N^*)$.

The MFG theory was originally developed in a series of papers by Huang *et al.* [12–14], and independently by Lions and Lasry [16–18]. Furthermore, Huang [11] introduced a class of so-called major-minor agent MFG's, where all agents except a single major one have vanishing individual influence of order $1/N$, while the major agent, although dominant in terms of influence, has no particular priority in its decision making. Thus, the appropriate equilibrium concept is Nash. By contrast, Bensoussan *et al.* recently developed in [3, 4] a major-minor MFG model where the major agent plays *first* and only then are the minor agents allowed to make their decisions. In this setup, on which we build here, the agents seek a Stackelberg solution [2, 27].

Definition 2 Consider $N+1$ agents, a set of strategy profiles $S = S_0 \times \cdots \times S_N$, and for each agent k, a cost function $J_k(u_0, \ldots, u_N)$, $\forall (u_0, \ldots, u_N) \in S$. Suppose that agent 0 is the major agent. A strategy profile $(u_0^*, \ldots, u_N^*) \in S$ is called a Stackelberg solution with respect to the costs J_k, if there exists a map T from S_0 to $S_1 \times \cdots \times S_N$, such that for all $u_0 \in S_0$, $T(u_0)$ is a Nash Equilibrium with respect to J_k, $k = 1, \ldots, N$, and $u_0^* = \min_{u_0 \in S_0} J_0(u_0, T(u_0))$, with $T(u_0^*) = (u_1^*, \ldots, u_N^*)$.

Our work is also related to the literature on discrete choice models in microeconomics, which were initially developed to analyze human choice behavior in the face of a finite set of alternatives, e.g., the choice of a mode of transportation [15], of a residential location [5], smoking decision in schools [22], etc. Mcfadden [21] proposed the first static discrete choice model, where choices are dictated only by personal factors. Several models with social interactions were introduced later. For instance, Brock and Durlauf propose in [6], within the framework of static non-cooperative game theory, a model involving a large number of agents making a choice between two alternatives, while being influenced by the average choice of the population. Recently, we developed a dynamic discrete choice model in [25, 26], with no advertisement, which we analyzed using the MFG methodology.

The main contribution of this paper is to introduce an advertising model involving two competing alternatives and in which both the consumers and advertiser are part of the game. Our model describes the consumers' individual behaviors, from which it deduces the way the population of consumers splits along the alternatives under both the social and advertising effects. The mathematical model is formulated in Section 2. In Section 3, we consider the limiting infinite population case, and show that the advertiser can always make optimal investments if the consumers have sufficiently diverse a priori opinions towards the alternatives, i.e., a Stackelberg solution exists in that case. Section 4 considers the special case of consumers with a certain uniform distribution of a priori opinions and gives an explicit form of the Stackelberg equilibrium solution and of the final distribution of the consumers' choices over the alternatives. In Section 5, we discuss some illustrative numerical simulation results, while Section 6 presents our conclusions.

A preliminary version of our results appeared in the conference paper [28]. The proofs and many discussions of the results were omitted from the conference paper due to space limitations, and can be found here.

1.1 Notation

The following notation is used throughout the paper. The indicator function of a subset X is denoted by 1_X. The transpose of a matrix M is denoted by M'. We denote by I_k the $k \times k$ identity matrix. We denote the matrix product MM' by $M^{(2)}$.Throughout this paper, $L_2([0,T],\mathbb{R}^m)$ is endowed with the inner product $\langle f, g\rangle = \int_0^T f(t)'g(t)\mathrm{d}t$, and the corresponding norm is denoted by $\|.\|_2$.

2 Mathematical Model

The dynamic collective choice problem with an advertiser is modeled as a dynamic non-cooperative game involving N minor agents (mA) or consumers and one major agent (MA) or advertiser, with respective dynamics,

$$\dot{x}_i = Ax_i + Bu_i, \qquad \text{for } i = 1,\dots,N, \tag{1}$$

$$\dot{y} = A_0 y + B_0 v, \tag{2}$$

where $x_i \in \mathbb{R}^n$, x_i^0 and $u_i \in L_2([0,T],\mathbb{R}^m)$ are the state, initial state and control input of the mA i, while $y \in \mathbb{R}^{n_1}$, y^0 and $v \in L_2([0,T],\mathbb{R}^{m_1})$ are the state, initial state and control input of the MA. Throughout the paper, y will be referred to as the influence state. We assume that the initial conditions x_i^0, $1 \le i \le N$, are independent and identically distributed (i.i.d.) random vectors on some probability space $(\Omega, \mathcal{F}, P)$ with distribution P_0. In the remainder of the paper, $\mathbb{E}(X)$ denotes the expectation of a random variable X.

The mAs and MA are associated with the following individual cost functions:

$$J_i(u_i, \bar{x}, v) = \mathbb{E}\Bigg[\int_0^T \left\{ \frac{q}{2} \left\| x_i - \alpha\bar{x} - K(p_2)y \right\|^2 + \frac{r}{2}\left\| u_i \right\|^2 \right\} \mathrm{dt} \qquad (3)$$

$$+ \frac{M}{2} \min_{j=1,2} \left\| x_i(T) - p_j \right\|^2 \Bigg],$$

$$J_0(v, \bar{x}) = \mathbb{E}\Bigg[\int_0^T \frac{r_0}{2} \left\| v \right\|^2 \mathrm{dt} + \frac{M_0}{2} \left\| \bar{x}(T) - p_2 \right\|^2 \Bigg], \qquad (4)$$

for $i = 1, \ldots, N$, where $\bar{x} = \left(\sum_{i=1}^{N} x_i \right) / N$ is the average state, $\alpha \geq 0$, $q, r, r_0, M, M_0 > 0$, and $p_1, p_2 \in \mathbb{R}^n$ are the possible alternatives. The matrix $K(p_2) \in \mathbb{R}^{n \times n_1}$ is a function of the alternative p_2. M is a large number, which incites the final state of the mAs to be close to one of the two alternatives at time T.

Example 1 Consider a group of N teenagers choosing before a time T between smoking (p_1) or not smoking (p_2). At time t, teenager i's smoking inclination is modeled by a variable $x_i(t) \in [-1, 1]$, where the value -1 corresponds to a nonsmoker, while 1 represents a full smoker. The effort exerted by i at time t to alter its position on the smoking spectrum is captured by $u_i(t) \in \mathbb{R}$. For example, $|u_i|$ would represent the amount of money spent per unit time by i to intensify (buying extra cigarettes) or reduce (resorting to anti smoking treatment) its smoker status. On the other hand, the government rate of investments against smoking is modeled by a variable $v \in \mathbb{R}$. The variable y represents the *effectiveness* of the advertising investment. The influence exerted by the advertisement on the teenagers' smoking status is modeled by $K(p_2)y$, where $K(p_2) := p_2 =$ "Do Not Smoke" $= -1$. A teenager, in the process of choosing between not smoking or smoking, minimizes the cost (3), which penalizes along the path the deviation from the peers' average smoking status $\bar{x}$ and the government nonsmoking advertisement $K(p_2)y$, as well as the effort to change the smoking status. Moreover, the teenager should be by time T a smoker (p_1) or nonsmoker (p_2) lest he/she be considered indecisive by his/her peers. Thus, a lack of decision by time T is strongly penalized in the final cost. On the other hand, the government tries to minimize its advertisement investments (the running cost of (4)), and should convince by time T the teenagers to be nonsmokers. Failure to sway a majority of teenagers away from smoking results in a strong penalty in the final cost.

3 Mean Field Stackelberg Competition

In a Stackelberg competition, the MA plays first, and then the mAs make their decisions. The agents solve the game as follows. Given the MA strategy v, the mAs play a Nash equilibrium with respect to their individual costs (3). If for each v there exists a unique mA Nash equilibrium $(u_1^*(v),\dots,u_N^*(v))$, then the advertiser knows how the mAs respond to its investment strategies. In other words, it constructs a map that maps its strategies to the corresponding mA Nash equilibria. Then, the MA computes its optimal strategy by minimizing $J_0(v,\bar{x}_v)$, where $\bar{x}_v$ is the consumers' average state under the Nash equilibrium $(u_1^*(v),\dots,u_N^*(v))$.

In view of (3)-(4), the MA and each mAs interact with the mAs population only through the mean field term $\bar{x}$. An efficient methodology to solve dynamic games involving a large number of such weakly coupled agents is the MFG approach [14]. We start by assuming a continuum of mAs to which one can ascribe a deterministic but initially unknown mean trajectory $\bar{x}$. Then, the limiting game consists of (i) a representative (generic) mA of state x, control input u, and initial state x^0, where x^0 is a random vector of distribution P_0; and (ii) the MA defined in (2). The state x satisfies (1). In view of the assumed independence of the mAs' random initial conditions, the generic mA and MA limiting cost functionals are respectively,

$$J(u,\bar{x},v) = \int_0^T \left\{\frac{q}{2}\left\|x-\alpha\bar{x}-K(p_2)y\right\|^2+\frac{r}{2}\left\|u\right\|^2\right\}\mathrm{dt} + \frac{M}{2}\min_{j=1,2}\left\|x(T)-p_j\right\|^2, \tag{5}$$

$$\bar{J}_0(v,\bar{x}) = \int_0^T \frac{r_0}{2}\left\|v\right\|^2\mathrm{dt}+\frac{M_0}{2}\left\|\bar{x}(T)-p_2\right\|^2, \tag{6}$$

where $\bar{x}=\mathbb{E}[x]$. The costs (5)-(6) are those of the mA/MA, with the average of the mAs replaced by an assumed given deterministic trajectory $\bar{x}$.

In the following subsection, we show that there exists a mA Nash equilibrium for any MA strategy v. Moreover, we anticipate for each equilibrium the probability distribution of the mAs' choices over the alternatives. Indeed, we show that for each equilibrium, the fraction of consumers that choose p_1 under the social effect and advertising strategy v is a fixed point of a well defined map.

3.1 mA Nash Equilibrium

Given the MA strategy v associated with an influence state trajectory $t\mapsto y(t)$, we start by computing the generic mA's best response to $\bar{x}$. The cost function (5) can be written as the minimum of two Linear Quadratic Regulator (LQR) optimal tracking problems, each associated with one of the two alternatives. Hence, $J(u,\bar{x},v)=\min\Big(J^1(u,\bar{x},v),J^2(u,\bar{x},v)\Big)$, where

$$J^j(u,\bar{x},v) = \int_0^T \left\{\frac{q}{2}\left\|x-\alpha\bar{x}-K(p_2)y\right\|^2+\frac{r}{2}\left\|u\right\|^2\right\}\mathrm{dt}+\frac{M}{2}\left\|x(T)-p_j\right\|^2,$$

for $j=1,2$. As a result, the mA's best response is the optimal control law of the LQR problem with the least optimal cost. Accordingly, we define the *basin* *of attraction* $D(\bar{x}, y)$ as the set of initial conditions for a generic mA such that the LQR optimal tracking problem corresponding to p_1 is the less costly, and hence the generic agent goes towards p_1 by applying the optimal control law of J^1. A generic mA whose initial condition does not belong to $D(\bar{x}, y)$ moves towards p_2 by applying the optimal control law of J^2. Designating respectively by u^* and n the best response and the associated costate of a generic mA, the Maximum Principle leads to the following equations:

$$\begin{aligned} u^* &= -\frac{1}{r} B' n \\ -\dot{n} &= A' n + q\left(x - \alpha \bar{x} - K(p_2) y\right), \end{aligned} \tag{7}$$

with $n(T) = M\left(x(T) - p_1 1_{D(\bar{x},y)}(x^0) - p_2 1_{D(\bar{x},y)^c}(x^0)\right)$. Equations (7) follow from the fact that if $x^0 \in D(\bar{x}, y)$, then the optimal control law u^* is the optimal control law of J^1, which is equal to $-\frac{1}{r} B' n^1$. Here, n^1 is the optimal costate of J^1, which satisfies the costate equation in (7) with the boundary condition $n^1(T) = M(x(T) - p_1)$. Similarly when $x^0 \in D(\bar{x}, y)^c$. Noting that both best response costs are quadratic functions of the initial state x^0, with only the linear and constant terms differing as they respectively depend on the trajectory to be tracked $(\alpha\bar{x} + K(p_2)y)$ and the chosen alternative in $J^j(u, \bar{x}, v)$, $j = 1, 2$, it is possible to show that:

$$\begin{aligned} D(\bar{x}, y) &:= \left\{x^0 \in \mathbb{R}^n | J^1_*(x^0, \bar{x}, v) \leq J^2_*(x^0, \bar{x}, v)\right\} \\ &= \left\{x^0 \in \mathbb{R}^n | \beta' x^0 \leq \delta + \Delta\left(\alpha \bar{x} + K(p_2) y\right)\right\}, \end{aligned}$$

where $J^j_*(x^0, \bar{x}, v)$ is the optimal cost of the LQR optimal tracking problem associated with p_j, Δ is a linear form on $L_2([0, T], \mathbb{R}^n)$, β and δ are constants, with the expressions of Δ, β and δ given in the Appendix.

Given the macroscopic behavior $\bar{x}$ and the MA influence state y, the generic mA's best response is thus uniquely determined. Now, for a given y trajectory, we study the existence of a *consistent* macroscopic behavior $\bar{x}$, i.e., one such that $\bar{x}$ is indeed the mean of the mAs' states when optimally responding to the $\bar{x}$, y dependent tracking trajectories in the cost functions $J^j(u, \bar{x}, v)$, $j = 1, 2$. This essentially corresponds to *a fixed point property* of the admissible $\bar{x}$ trajectories, which must be self replicating as the means of the above $\bar{x}$, y dependent agents' best response trajectories. By taking the expectations of the right and left hand sides of (1) and (7), and in view of the linear dependence on initial and final conditions, we obtain that $\bar{x}$ must satisfy the following Mean Field equation system (MF)

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} - \frac{1}{r} B^{(2)} n \\ -\dot{\bar{n}} &= A'\bar{n} + q(1 - \alpha)\bar{x} - qK(p_2) y, \end{aligned} \tag{8}$$

with $\bar{x}(0) = \mathbb{E}x^0 := \mu_0$, $\bar{n} = \mathbb{E}n$, $\bar{n}(T) = M(\bar{x}(T) - p_\lambda)$, $\lambda = P_0\Big(D(\bar{x}, y)\Big)$, and $p_\lambda = \lambda p_1 + (1 - \lambda) p_2$. Note that λ is the fraction of mAs that go towards p_1. We make the following technical assumption.

Assumption 1 *The following Riccati equation has a unique solution:*

$$\dot{\Pi} + \Pi A + A'\Pi - \frac{1}{r}\Pi B^{(2)}\Pi + q(1-\alpha)I_n = 0, \qquad \Pi(T) = MI_n. \tag{9}$$

Note that if $\alpha \leq 1$, then (9) has a unique solution [1, page 23]. For more details about the existence and uniqueness of solutions to (9), one can refer to [10]. Denoting Φ the unique solution of $\frac{d}{dt}\Phi(t,s) = \left(A - \frac{1}{r}B^{(2)}\Pi\right)\Phi(t,s)$, $\Phi(s,s) = I_n$, define $R(t) = \Phi(t,0)$ and the following quantities:

$$\bar{R}(t) = \frac{M}{r}\int_0^t \Phi(t,\sigma)B^{(2)}\Phi(T,\sigma)'\mathrm{d}\sigma, \tag{10}$$

$$\Xi(y)(t) = -\frac{q}{r}\int_0^t \int_T^\sigma \Phi(t,\sigma)B^{(2)}\Phi(\tau,\sigma)'K(p_2)y(\tau)\mathrm{d}\tau\mathrm{d}\sigma,$$

Moreover, let $F(\lambda, y) := P_0(H^\lambda(y))$, where

$$H^\lambda(y) = \left\{x^0 \in \mathbb{R}^n | \beta' x^0 \leq \delta + \Delta\left(K(p_2)y + \alpha\left(R\mu_0 + \bar{R}p_\lambda + \Xi(y)\right)\right)\right\}. \tag{11}$$

In the following lemma, we show that there exists a one to one map between the solutions $\bar{x}$ of the MF equation systems (8) and the fixed points of the finite dimensional function $\lambda \mapsto F(\lambda, y)$. The existence of the latter is guaranteed under the following assumption.

Assumption 2 *We assume that P_0 is such that the P_0-measure of hyperplanes is zero.*

Using techniques similar to those used in [26, Theorem 6], one can show the following Lemma.

Lemma 1 *Under Assumptions 1 and 2, the following statements hold:*

1. *$\bar{x}(t)$ is a solution of the MF equation system* (8) *if and only if it can be written under the form:*

$$\bar{x}(t) = \bar{x}^\lambda := R(t)\mu_0 + \bar{R}(t)p_\lambda + \Xi(y)(t), \tag{12}$$

   where $\lambda = F(\lambda, y)$, i.e., λ is fixed point of $\lambda \mapsto F(\lambda, y)$ defined above (11).
2. *The function $\lambda \mapsto F(\lambda, y)$ has at least one fixed point. Equivalently the MF equation system* (8) *has at least one solution.*

To prove the first point, we consider λ at first as a parameter. In this case, (8) is a linear forward-backward differential equation parameterized by λ. Under Assumption 1, $\bar{n}$ can be written as an affine function of $\bar{x}$, i.e., $\bar{n}(t) = \Pi(t)\bar{x}(t) + \gamma(t)$, where γ is the unique solution of $\frac{d}{dt}\gamma = -(A - \frac{1}{r}B^{(2)}\Pi)'\gamma - qK(p_2)y$, $\gamma(T) = -Mp_\lambda$. By replacing this form of $\bar{n}$ in (8), one can show that $\bar{x}$ is equal to (12). Thus, a fixed point path $\bar{x}$ is of the form (12), where $\lambda = P_0(D(\bar{x}, y)) = P_0(D(R(t)\mu_0 + \bar{R}(t)p_\lambda + \Xi(y)(t), y)) = P_0(H^\lambda(y)) = F(\lambda, y)$. Hence, λ is a fixed point of $\lambda \mapsto F(\lambda, y)$, for y fixed. The converse is proved by a simple verification argument.The existence of a fixed point in 2. is established by capitalizing on the finite dimensional nature of operator $\lambda \mapsto F(\lambda, y)$ to rely on Brouwer's fixed point theorem.

To summarize, the mAs make their choices of alternatives under the social effect and advertising investment v as follows. They compute a fixed point λ of $\lambda \mapsto F(\lambda, y)$, and the corresponding mean trajectory $\bar{x}$ defined in (12). If a consumer is initially in $D(\bar{x}, y)$, then it goes towards p_1. Otherwise, it moves towards p_2. In an infinite population, the best responses u^* (7) to $\bar{x}$ constitute a Nash equilibrium for the mAs. But in practice, this assumption makes these strategies less robust in face of unilateral deviant behaviors when applied to a finite number of consumers. This loss of performance is however negligible when N is large enough. Indeed, following the argument in [26, Theorem 9], one can show that for a finite population of N mAs, the strategies u^* defined by (7) for any MA strategy v is an $\epsilon-$Nash equilibrium with respect to the costs J_i, $i = 1, \dots, N$ defined in (3), where ϵ goes to zero as N increases to infinity.

Having established in Lemma 1 the existence of at least one infinite population Nash equilibrium for an arbitrary MA strategy v, we now turn to the problem of the MA itself optimizing its influence function in a Stackelberg fashion. To compute its optimal strategy, the MA should be able, for each strategy v, to anticipate *uniquely* the mA Nash equilibrium. Next, we present a condition under which a unique mA equilibrium exists.

Assumption 3 *We assume that* $\bar{F}(s) := P\left(\beta' x^0 \leq \delta + s\right)$ *is differentiable and* $\left|\frac{d}{ds}\bar{F}(s)\right| < \frac{1}{|\alpha \Delta\left(\bar{R}(p_1 - p_2)\right)|}$ *(here the linear form* Δ *defined in (24) acts on the function* $\bar{R}(t)(p_1 - p_2)$*).*

Noting that the function $\frac{d}{ds}\bar{F}$ is the probability density function of $\beta' x^0 - \delta$, Assumption 3 requires that the consumers' a priori opinions x^0 in the direction β have enough spread. For example, if the consumers' a priori opinions x^0 are distributed according to the normal distribution $\mathcal{N}(\mu_0, \Sigma_0)$, then $\beta' x^0 - \delta$ is distributed according to $\mathcal{N}(\beta' \mu_0 - \delta, \beta' \Sigma_0 \beta)$, and the corresponding probability density function has a maximum $1/\left(\sqrt{2\pi \beta' \Sigma_0 \beta}\right)$. In this case, Assumption 3 is satisfied if $2\pi \beta' \Sigma_0 \beta > \left(\alpha \Delta\left(\bar{R}(p_1 - p_2)\right)\right)^2$. Under Assumption 3, the function $\lambda \mapsto F(\lambda, y)$ is a contraction. Indeed, $\frac{d}{d\lambda} F = \alpha \Delta\left(\bar{R}(p_1 - p_2)\right) \frac{d}{ds}\bar{F}$, which under Assumption 3 has an absolute value strictly less than one. Therefore, we can state the following theorem.

Theorem 1 *Under Assumptions 1, 2, and 3, given the MA strategy* v *and resulting trajectory* y*,* $\lambda \mapsto F(\lambda, y)$ *has a unique fixed point. Thus, the mA limiting game admits a unique Nash equilibrium.*

3.2 MA Optimal Control Problem

Having determined the consumers' individual and macroscopic (mean trajectory) responses to the investment strategies, we turn now to the problem of finding an optimal investment policy v^*. We assume in the rest of this section that Assumptions 1, 2 and 3 hold to guaranty the existence of a unique Nash equilibrium for each $v \in L_2([0,T])$. The solution of the MF equation system (8) that corresponds to v is denoted by $\bar{x}_v$. It is the mean trajectory of the consumers under their best responses to it, u^* defined in (7). Thus, the advertiser solves the following optimal control problem:

$$\min_{v \in L_2([0,T])} \bar{J}_0\Big(v, \bar{x}_v\Big)$$
$$\text{s.t. } \dot{y} = A_0 y + B_0 v \text{ and } \dot{\bar{x}}_v = A\bar{x}_v + B\mathbb{E}u^*. \tag{13}$$

In the following theorem, we show that if the consumers' a priori opinions are sufficiently diverse, then the advertiser can find an optimal investment policy v^*. Afterwards, we characterize in Theorem 3 this strategy as the costate of $(y_{v^*}, \bar{x}_{v^*})$, where y_{v^*} is the advertiser's optimal state that corresponds to v^*. This allows us to derive explicit optimal investment policies in some situations, for example, in case the consumers' initial opinions are uniformly distributed in the direction β (See Section 4 below). The proofs of theorems and lemmas are given in the Appendix.

Theorem 2 *Under Assumptions 1, 2 and 3, the MA optimal control problem (13) has an optimal control law v^*.*

In the following theorem, we characterize an optimal strategy v^* as the costate of $(\bar{x}_{v^*}, y_{v^*})$. Given an optimal control law v^*, we define the costate equations:

$$-\dot{P} = A_0' P + \mathcal{L}_1^*(Q)(t) \tag{14}$$
$$-\dot{Q} = \mathcal{L}_2^*(Q)(t) \tag{15}$$

with $P(T) = 0$ and $Q(T) = M_0(\bar{x}_{v^*}(T) - p_2)$, where for all $z \in L_2([0,T], \mathbb{R}^n)$,

$$\mathcal{L}_1^*(z)(t) = \frac{q}{r} K(p_2)' \int_0^t \Phi(t,\sigma) B^{(2)} z(\sigma) \mathrm{d}\sigma + \xi^* K(p_2)' H(t) \int_0^T \Phi(T,\sigma) B^{(2)} z(\sigma) \mathrm{d}\sigma$$
$$\mathcal{L}_2^*(z)(t) = \left(A - \frac{1}{r} B^{(2)} \Pi\right)' z(t) + \xi^* \alpha H(t) \int_0^T \Phi(T,\sigma) B^{(2)} z(\sigma) \mathrm{d}\sigma, \tag{16}$$
$$\text{with } \xi^* = \frac{d\bar{F}}{ds} \left(\Delta \left(\alpha \bar{x}_{v^*} + K(p_2) y_{v^*}\right)\right).$$

Here, $H(t) = \frac{M^2 q}{r^2} \int_0^t \phi(\eta,t)' B^{(2)} \phi(\eta,T) \mathrm{d}\eta (p_1 - p_2)^{(2)}$, and ϕ is defined in the Appendix.

Theorem 3 *Under Assumptions 1, 2, and 3, if v^* is an optimal control law of (13) and the corresponding equations (14)-(15) have a unique solution (P, Q), then*

$$v^* = -\frac{1}{r_0} B_0' P. \tag{17}$$

Theorem 2 states that the advertiser can act optimally. But, it doesn't give any indication on how to compute the optimal investment strategies v^*. Theorem 3, however, provides a formula of v^* (17). As a result, the computation of v^* requires solving the advertiser's state equation (2) and the mAs MF equations (8), which are coupled with the costate equations (14)-(15) through the optimal control law (17). These equations show that the advertiser needs only to know the probability distribution of the consumers' a priori opinions P_0 to make optimal investments v^*. Even though the MA doesn't know the exact initial states of the consumers and their individual choices, it can anticipate the fraction of the mAs that go toward each alternative. Indeed, once v^* is computed, the fraction of mAs that choose p_1 is the unique fixed point of $\lambda \mapsto F(\lambda, y_{v^*})$. In Section 4, we study a special case where the optimal strategies v^* can be computed explicitly.

Before moving to the next section, we give a sufficient condition for the existence and uniqueness of solutions to (14)-(15) to hold. This condition is needed to apply the results of Theorem 3 later. Given the function Q, equation (14) is a linear differential equation which has a unique solution. Thus, it is sufficient to study the second equation (15). We define the matrix

$$\Sigma = \alpha \int_0^T \int_\sigma^T \Big(\Phi(T,\sigma)B\Big)^{(2)} \Phi(\tau,T)' H(\tau) \mathrm{d}\tau \mathrm{d}\sigma. \tag{18}$$

Assumption 4 *Either ξ^* is equal to zero or $1/\xi^*$ is not an eigenvalue of Σ, where ξ^* is defined in* (16).

Assumption 4 can be satisfied, for example, in the following two cases:

1. If the initial spread of the mAs is sufficient ($d\bar{F}/ds$ is low enough).
2. If $d\bar{F}/ds$ is bounded, and T is small enough.

In fact, $\xi^*\Sigma$ is in both cases negligible with respect to I_n. Hence, $1/\xi^*$ is not an eigenvalue of Σ.

Lemma 2 *Under Assumption 4, (15) has a unique solution.*

4 Case of Uniform Initial Distribution

Because ξ^* defined in (16) is a nonlinear functional of $\bar{x}_{v^*}$ and y_{v^*}, solving (2)-(8)-(14)-(15) is not easy. Note however that $d\bar{F}/ds$ is the probability density function of $\beta' x^0 - \delta$ (see the definition of $\bar{F}$ in Assumption 3), so one can hope to compute an explicit solution when this random variable is uniformly distributed, for example. Indeed, in this case the probability density function is piecewise constant. Hence, equations (2)-(8)-(14)-(15) can be written as a pair of forward-backward linear ordinary differential equations (19). These equations are coupled in the boundary condition K_λ, through λ the probability that a generic mA is initially in $D(\bar{x}_{v^*}, y_{v^*})$. Thus, they have similar structure to the mA MF equations (8). Here again, we use similar techniques to those used in [26, Theorem 6] to provide an explicit solution to (2)-(8)-(14)-(15), see

Theorem 5 below. This solution encapsulates the advertiser's optimal investment strategy v^* and influence function y_{v^*}, the mA mean trajectory $\bar{x}_{v^*}$, as well as the fraction of mAs that go towards p_1 under a social and advertising effects.

So, we assume in this section that the mAs' initial states are uniformly distributed in the direction β. More precisely, we assume that $\beta' x^0 - \delta$ has a uniform distribution $U([a-c/2, a+c/2])$, where $a \in \mathbb{R}$ and $c > 0$. We show in this case that if the initial spread of the mAs is sufficient (see Assumption 5 below), then there exists a unique Stackelberg solution. It should be noted that Assumption 2 is satisfied for this distribution of initial states.

The function $\bar{F}$ is piecewise linear, hence only piecewise differentiable. Therefore, we need an alternative to Assumption 3, under which the uniqueness of the mA Nash equilibria holds. Moreover, in order to apply the variational methods of Subsection 3.2, we require $\bar{F}$ to stay in a differentiable domain for all the MA strategies, which is the case when the mA are sufficiently spread (see Lemma 3 below).

Assumption 5 *We assume that* $c > \alpha \left| \Delta\big(\bar{R}(p_1 - p_2)\big) \right|$.

Under Assumption 5, given the MA strategy v, the mA limiting game admits a unique Nash equilibrium by virtue of Theorem 2.

Theorem 4 *Under Assumptions 1 and 5, the MA optimal control problem (13) has an optimal control law* v^*.

Lemma 3 *Under Assumptions 1 and 5, there exists* $c_0 > 0$ *independent of* v *such that for all* $c > c_0$*, there exists a unique mA Nash equilibrium corresponding to* $\lambda \in (0,1)$.

For the rest of the analysis, we assume that $c > c_0$. In this case, the unique fixed point λ corresponding to a MA optimal control law v^* is in $(0,1)$. Since F is differentiable in $(0,1)$, one can use techniques similar to those used in Theorem 3 to show that v^* satisfies (17), provided that the Assumptions 1 and 5 are satisfied, and $1/c$ is not an eigenvalue of Σ defined in (18).

In the following, we write the equations (2)-(8)-(14)-(15) as a pair of forward-backward differential equations. To this end, we define the states $h = (\bar{x}_{v^*}, y_{v^*}, q_1)$, $d = (\bar{n}, P, Q, q_2)$. Here, $q_1(t) := \int_0^t \Phi(T,\sigma) B^{(2)} Q(\sigma) \mathrm{d}\sigma$ and $q_2(t) := \int_t^T \Phi(T,\sigma) B^{(2)} Q(\sigma) \mathrm{d}\sigma$ are respectively the forward and backward propagating parts of $\int_0^T \Phi(T,\sigma) B^{(2)} Q(\sigma) \mathrm{d}\sigma$, which appears in (14)-(15). The pair (h, d) satisfies

$$\begin{aligned} \dot{h} &= K_1(t) h + K_2(t) d \\ \dot{d} &= K_3(t) h + K_4(t) d \end{aligned} \tag{19}$$

with $h(0) = h_0 = (\mu_0, y^0, 0)$ and $d(T) = K_5 h(T) + K_\lambda$, where $K_1(t) = \mathrm{diag}(A, A_0, 0)$,

$$K_2(t) = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \qquad K_5 = \begin{bmatrix} MI_n & 0 & 0 \\ 0 & 0 & 0 \\ M_0 I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$K_3(t) = \begin{bmatrix} -q(1-\alpha)I_n & qK(p_2) & 0 \\ 0 & 0 & k_4 \\ 0 & 0 & -\frac{\alpha H(t)}{c} \\ 0 & 0 & 0 \end{bmatrix}, K_4(t) = \begin{bmatrix} -A' & 0 & 0 & 0 \\ 0 & -A_0' & 0 & -\frac{K(p_2)' H(t)}{c} \\ 0 & 0 & k_5 & -\frac{\alpha H(t)}{c} \\ 0 & 0 & -k_3 & 0 \end{bmatrix},$$

$K_\lambda = -\Big(Mp_\lambda, 0, M_0 p_2, 0\Big)$, $k_1 = -\frac{1}{r}B^{(2)}$, $k_2 = -\frac{1}{r_0}B_0^{(2)}$, $k_3 = \Phi(T,t)B^{(2)}$, $k_4 = -K(p_2)'\Big(\frac{q}{r}\Phi(t,T) + \frac{H(t)}{c}\Big)$ and $k_5 = -(A - \frac{1}{r}B^{(2)}\Pi)'$.

The equation system (19) consists of two coupled forward-backward differential equations. The final condition $d(T)$ depends through λ non-linearly on the path $(\bar{x}_{v^*}(\sigma), y_{v^*}(\sigma))$, $\sigma \in [0,T]$. As in Lemma 1, we need the following assumption to decouple and solve these equations. This assumption plays the role of Assumption 1 in Lemma 1.

Assumption 6 *The following generalized Riccati equation has a unique solution*

$$\dot{W} = K_4 W - WK_1 - WK_2 W + K_3, \qquad W(T) = K_5. \tag{20}$$

Here again, one can use similar techniques to those used in [26, Theorem 6] to show that under Assumption 6, (h, d) is a solution of (19) if and only if

$$h(t) = \Phi_1(t,0)h_0 + R_u(t)K_\lambda := \left(\bar{x}^\lambda, y^\lambda, q_1^\lambda\right), \tag{21}$$
$$d(t) = W(t)h(t) + \Phi_2(t,T)K_\lambda := \left(\bar{n}^\lambda, P^\lambda, Q^\lambda, q_2^\lambda\right) \tag{22}$$

where Φ_1 and Φ_2 are the unique solutions of $\frac{d}{dt}\Phi_1(t,s) = (K_1 + K_2 W)\Phi_1(t,s)$, $\Phi_1(s,s) = I_{2n+n_1}$ and $\frac{d}{dt}\Phi_2(t,s) = (K_4 - WK_2)\Phi_2(t,s)$, $\Phi_2(s,s) = I_{3n+n_1}$, $R_u(t) = \int_0^t \Phi_1(t,\sigma)K_2(\sigma)\Phi_2(\sigma,T)\mathrm{d}\sigma$ and λ is a fixed point of the following final dimensional map,

$$F_u(\lambda) = \bar{F} \circ \Delta\left(\alpha\bar{x}^\lambda + K(p_2)y^\lambda\right). \tag{23}$$

For a brief discussion of the proof, we refer the reader to the discussions below Lemma 1.

Theorem 5 *Under Assumptions 1, 5 and 6, the Stackelberg competition (for a continuum of consumers) has a unique solution* $(v^*, \bar{x}_{v^*})$*, where* $v^* = -\frac{1}{r_0}B_0' P^{\lambda_*}$ *and* $\bar{x}_{v^*} = \bar{x}^{\lambda_*}$*, with* P^{λ_*} *and* $\bar{x}^{\lambda_*}$ *defined in* (21)-(22) *for the unique fixed point* λ_* *of* F_u*.*

Theorem 5 states that for a population of consumers with sufficiently diverse a priori opinions uniformly distributed in the direction β, an advertiser can design a unique optimal investment strategy $v^* = -\frac{1}{r_0} B_0' P^{\lambda_*}$. This strategy convinces a fraction of the consumers to choose the advertised alternative. This fraction is equal to $1 - \lambda_*$, where λ_* is the unique fixed point of F_u. One can apply the bisection method to find λ_*. Once λ_* is computed, the agents can compute the vectors (h, d), given by (21)-(22). The MA can then implement its optimal strategy (17), where P^{λ_*} is the second component of d. Subsequently, the mAs can predict their limiting macroscopic behavior, the first component of d, and implement their optimal strategies (7).

5 Simulations

To illustrate the collective choice mechanism in the presence of social and advertising effects, we consider a group of 6000 consumers that have opinion states initially uniformly distributed between -25 and 5. The consumers are choosing between $p_1 = -20$ and $p_2 = 20$. The social effect is represented by $\alpha\bar{x}$, where $\alpha = 0.5$. We consider two scenarios. In the first one, the consumers make their choices in the absence of an advertising effect ($K(p_2) = 0$), while in the second scenario, an advertiser advertises for p_2. The advertising effect is modeled in the cost by $K(p_2)y = p_2 y$, where y is the (influence) state of the MA. We set $T = 3$, $A = 0.5$, $B = 0.1$, $A_0 = -0.1$, $B_0 = 0.1$, $y_0 = 0$, $q = 10$, $r = r_0 = 10$, and $M = M_0 = 2000$. In the absence of an advertising effect, $\lambda_* = 0.84$ is the unique fixed point of F_u defined in (23). Accordingly, 84% of the mAs go towards p_1 (Fig. 1). On the other hand, with advertisement for alternative p_2, the fraction of mAs that go towards p_2 increases from 16% to 87%, see Fig. 1.

6 Conclusion

We introduce in this paper a dynamic collective choice model in the presence of social and advertising effects. In this model, a large group of consumers choose between two alternatives while influenced by their average and an advertising effect. The latter is exerted by a Stackelbergian advertiser aiming at convincing the population of consumers to choose p_2. We consider the limiting infinite population game and derive conditions under which a Stackelberg solution exists. In case the consumers' a priori opinions are distributed uniformly in a specific direction, we give an explicit form of the unique Stackelberg solution, and determine the fraction of minor agents that choose p_1. This fraction is the unique fixed point of well defined map, and can be computed by knowing the a priori opinions' distribution. Finally, it is of interest for future work to extend the results to the case of multiple competitive advertisers, with more than two alternatives.

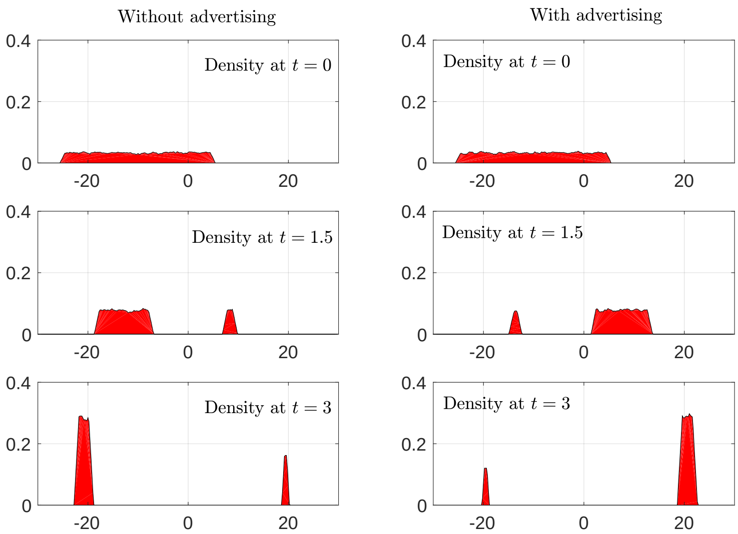

Fig. 1 Evolution of the consumers' density in the absence and presence of advertising.

Appendix A

Quantities related to the mA Nash equilibrium:

For all $x \in L_2([0,T],\mathbb{R}^n)$,

$$\Delta(x) = \frac{Mq}{r}(p_1 - p_2)' \int_T^0 \int_T^\eta \phi(\eta,T)' B^{(2)} \phi(\eta,\sigma) x(\sigma)\, \mathrm{d}\sigma \mathrm{d}\eta, \qquad (24)$$

where ϕ is the unique solution of $\frac{d}{dt}\phi(t,s) = \left(\frac{1}{r}\Gamma(t)B^{(2)} - A'\right)\phi(t,s)$, $\phi(s,s) = I_n$, and

$$\begin{aligned}
\dot{\Gamma} &= \frac{1}{r}\Gamma B^{(2)}\Gamma - \Gamma A - A'\Gamma - qI_n, \qquad \Gamma(T) = MI_n \\
\beta &= M\phi(0,T)(p_2 - p_1) \\
\delta &= \frac{1}{2}M(\|p_2\|^2 - \|p_1\|^2) + \frac{M^2}{2r}p_2' \int_T^0 \left(\phi(\eta,T)'B\right)^{(2)} \mathrm{d}\eta\, p_2 \\
&\quad - \frac{M^2}{2r}p_1' \int_T^0 \left(\phi(\eta,T)'B\right)^{(2)} \mathrm{d}\eta\, p_1.
\end{aligned}$$

Proof of Theorem 2:

The cost functional $\bar{J}_0$ is positive and coercive with respect to $v \in L_2([0,T])$, i.e., $\lim_{\|v\|_2 \to \infty} \bar{J}_0(v)/\|v\|_2 = \infty$. If we show that $\bar{J}_0$ is continuous in the reflexive

Banach space $L_2([0,T])$ with respect to v, then by Tonelli's existence theorem [7, Theorem 5.51], $\bar{J}_0$ has a finite minimum. Thus, we need only show that $\bar{J}_0$ is continuous. The state y is continuous with respect to v. The fixed points $\lambda(y)$ of F are continuous with respect to v. In fact, consider v and v' in $L_2([0,T])$ and denote by y, y' the corresponding MA trajectories and by λ and λ' the corresponding fixed points. We have

$$\begin{aligned}|\lambda-\lambda'| &= |F(\lambda,y)-F(\lambda',y')| \leq |F(\lambda,y)-F(\lambda',y)| + |F(\lambda',y)-F(\lambda',y')| \\ &\leq \sup_{s\in[0,1]}\left|\frac{dF}{d\lambda}(s,y)\right||\lambda-\lambda'| + |F(\lambda',y)-F(\lambda',y')|.\end{aligned}$$

Therefore, $\left(1-\sup_{s\in[0,1]}\left|\frac{dF}{d\lambda}(s,y)\right|\right)|\lambda-\lambda'| \leq |F(\lambda',y)-F(\lambda',y')|$. Under Assumption 3, $\sup_{s\in[0,1]}\left|\frac{dF}{d\lambda}(s,y)\right| < 1$. Under Assumption 2, $\bar{F}$ is continuous. Moreover, Δ is continuous with respect to the L_2 norm $\|\|_2$. Hence, F is continuous with respect to y, and $|F(\lambda',y)-F(\lambda',y')|$ converges to zero as $\|y-y'\|_2$ converges to zero. Therefore, the fixed points λ of F are continuous. In view of (12) and the continuity of the fixed points λ, $\bar{x}(T)$ is continuous. Therefore, $\bar{J}_0$ is continuous.

Proof of Theorem 3:

We derive the condition on v^* (17) by studying the first variation of the cost functional in (13) with respect to a perturbation $v = v^* + \eta\delta v$, where $\eta \in \mathbb{R}$, and $\delta v \in L_2([0,T],\mathbb{R}^{m_1})$. To this end, we need to derive at first an explicit form of the constraint on $\bar{x}_v$. We have that $\bar{x}_v = \bar{x}^\lambda$ defined in (12), where λ is the unique fixed point of $\lambda \mapsto F(\lambda,y)$. By taking the derivative of $\bar{x}^\lambda$ with respect to time, we obtain that,

$$\dot{\bar{x}}_v = \mathcal{L}(\bar{x}_v,y)(t), \qquad \bar{x}(0) = \mu_0, \tag{25}$$

where

$$\begin{aligned}\mathcal{L}(\bar{x}_v,y)(t) &= \left(A - \frac{1}{r}B^{(2)}\Pi\right)\bar{x}_v - \frac{q}{r}B^{(2)}\int_T^t \Phi(\sigma,t)'K(p_2)y(\sigma)\mathrm{d}\sigma \\ &+ \frac{M}{r}B^{(2)}\Phi(T,t)'\bar{F}\circ\Delta\big(\alpha\bar{x}_v + K(p_2)y)\big)(p_1-p_2) + \frac{M}{r}B^{(2)}\Phi(T,t)'p_2.\end{aligned}$$

We compute now the Gâteaux derivatives [7] of y and $\bar{x}$ at v^* in the direction δv:

$$\begin{aligned}\frac{d}{d\eta}y_{v^*+\eta\delta v}\Big|_{\eta=0} &:= \delta y \\ \frac{d}{d\eta}\bar{x}_{v^*+\eta\delta v}\Big|_{\eta=0} &:= \delta\bar{x},\end{aligned} \tag{26}$$

where,

$$
\begin{aligned}
&\frac{d}{dt}\delta y = A_0 \delta y + B_0 \delta v, \quad \delta y(0) = 0 \\
&\frac{d}{dt}\delta \bar{x} = \mathcal{L}_1(\delta y)(t) + \mathcal{L}_2(\delta \bar{x})(t), \quad \delta \bar{x}(0) = 0,
\end{aligned}
$$

and $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) is a continuous linear operator from the Hilbert space $L_2([0,T],\mathbb{R}^{n_1})$ (resp. $L_2([0,T],\mathbb{R}^{n})$) to $L_2([0,T],\mathbb{R}^{n})$ such that for all $z_1 \in L_2([0,T],\mathbb{R}^{n_1})$ and $z_2 \in L_2([0,T],\mathbb{R}^{n})$,

$$
\begin{aligned}
\mathcal{L}_1(z_1)(t) &= -\frac{q}{r} B^{(2)} \int_T^t \Phi(\sigma,t)' K(p_2) z_1(\sigma) \mathrm{d}\sigma \\
&\quad + \frac{M}{r} \xi^* \Delta \left(K(p_2) z_1 \right) B^{(2)} \Phi(T,t)'(p_1 - p_2), \\
\mathcal{L}_2(z_2)(t) &= \left(A - \frac{1}{r} B^{(2)} \Pi \right) z_2(t) + \frac{M\alpha}{r} \xi^* \Delta \left(z_2 \right) B^{(2)} \Phi(T,t)'(p_1 - p_2).
\end{aligned}
$$

Using Fubini-Tonelli's theorem [23], one can show that the adjoint operators of $\mathcal{L}_1$ and $\mathcal{L}_2$ are respectively $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ defined in (16). We recall from [24] that the adjoint operator of a linear continuous operator $\mathcal{G}$ defined from the Hilbert space $(H_1, \langle , \rangle_1)$ into the Hilbert space $(H_2, \langle , \rangle_2)$ is the linear continuous operator $\mathcal{G}^*$ defined from the Hilbert space $(H_2, \langle , \rangle_2)$ into the Hilbert space $(H_1, \langle , \rangle_1)$ and satisfying for all $x \in H_1$ and $y \in H_2$ $\langle \mathcal{G}(x), y \rangle_2 = \langle x, \mathcal{G}^*(y) \rangle_1$. Here, we use the explicit form of the operator Δ (24). The Gâteaux derivative of $\bar{J}_0$ is

$$
\begin{aligned}
\delta \bar{J}_0 &= \frac{d}{d\eta} \bar{J}_0 \Big(v^* + \eta \delta v, \bar{x}_{v^* + \eta \delta v} \Big) \Big|_{\eta = 0} \\
&= r_0 \Big\langle v^*, \delta v \Big\rangle + M_0 (\bar{x}_{v^*}(T) - p_2)' \delta \bar{x}(T).
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{d}{dt}(\delta y' P) = \delta v' B_0' P - \delta y' \mathcal{L}_1^*(Q)(t) \\
&\frac{d}{dt}(\delta \bar{x}' Q) = \mathcal{L}_1(\delta y)(t)' Q + \mathcal{L}_2(\delta \bar{x})(t)' Q - \delta \bar{x}' \mathcal{L}_2^*(Q)(t).
\end{aligned} \tag{27}
$$

By integrating (27) from 0 to T we get $0 = \Big\langle B_0' P, \delta v \Big\rangle - \Big\langle \mathcal{L}_1^*(Q)(t), \delta y \Big\rangle$. Similarly, we have

$$
\begin{aligned}
M_0 \delta \bar{x}(T)' (\bar{x}_{v^*}(T) - p_2) &= \Big\langle \mathcal{L}_1(\delta y)(t), Q \Big\rangle + \Big\langle \mathcal{L}_2(\delta \bar{x})(t), Q \Big\rangle - \Big\langle \delta \bar{x}, \mathcal{L}_2^*(Q)(t) \Big\rangle \\
&= \Big\langle \mathcal{L}_1^*(Q)(t), \delta y \Big\rangle.
\end{aligned}
$$

Therefore, $\delta \bar{J}_0 = \Big\langle B_0' P, \delta v \Big\rangle + r_0 \Big\langle v^*, \delta v \Big\rangle$. By optimality, $\delta \bar{J}_0 = 0$ for all $\delta v \in L_2([0,T])$. Hence, $v^* = -\frac{1}{r_0} B_0' P$.

Proof of Lemma 2:

The idea of the proof is to replace the term

$$\int_0^T \Phi(T,\sigma)B^{(2)}Q(\sigma)\mathrm{d}\sigma \tag{28}$$

in the expression of $\mathcal{L}_2^*(Q)$ by an assumed known constant K_1. Equation (15) is then a linear differential equation parameterized by K_1 whose solution is a linear operator of K_1. By replacing this solution in the term (28), and by requiring that K_1 is equal to (28), one can show that the unique solution of (15) is $Q(t) = \Phi(T,t)'\Big(\alpha\xi^* \int_t^T \Phi(\sigma,T)'H(\sigma)\mathrm{d}\sigma Y + M_0(\bar{x}_{v^*}(T) - p_2)\Big)$, where Y is the unique solution of the following linear algebraic equation:

$$(I_n - \xi^* \Sigma)Y = M_0 \int_0^T \Big(\Phi(T,\sigma)B\Big)^{(2)} \mathrm{d}\sigma(\bar{x}^*(T) - p_2).$$

Proof of Theorem 4:

Let v and v' in $L_2([0,T])$, and denote by y, y' the corresponding MA trajectories, and by λ and λ' the corresponding fixed points. We have,

$$\begin{aligned}|\lambda - \lambda'| = |F(\lambda,y) - F(\lambda',y')| &\leq |F(\lambda,y) - F(\lambda',y)| + |F(\lambda',y) - F(\lambda',y')| \\ &\leq \frac{\alpha}{c}\Big|\Delta\big(\bar{R}(t)(p_1 - p_2)\big)\Big||\lambda - \lambda'| + |F(\lambda',y) - F(\lambda',y')|.\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.

Proof of Lemma 3:

The uniqueness follows from Assumption 5. Let $v \in L_2([0,T])$. The path $\bar{x}_v$ defined in (12) is uniformly bounded with c (with respect to the L_2 norm). Therefore, the optimal cost $\bar{J}_0(v^*, \bar{x}_{v^*}) \leq \bar{J}_0(v, \bar{x}_v)$ of the MA optimal control problem defined in (13) is uniformly bounded with c. Hence, the optimal control law v^* and the optimal state y_{v^*} are uniformly bounded with c. Consequently, the term $\Delta\left(K(p_2)y_{v^*} + \alpha\bar{x}^\lambda\right)$, where $\bar{x}^\lambda$ is defined in (12), is uniformly bounded with c by a positive constant L_1. This means that $-L_1 \leq \Delta\left(K(p_2)y_{v^*} + \alpha\bar{x}^\lambda\right) \leq L_1$. Hence, $\bar{F}(-L_1) \leq F(\lambda,y) \leq \bar{F}(L_1)$. If we choose $-L_1 > a - c/2$ and $L_1 < a + c/2$, that is, $c > \max(2(a+L_1), 2(-a+L_1)) := c_0$, then the map F takes its values in $(0,1)$. Therefore, F has a unique fixed point $\lambda \in (0,1)$.

Proof of Theorem 5:

By Theorem 4 we know that there exists v^* an optimal investment strategy. Moreover, we know that v^* and $\bar{x}_{v^*}$ should be equal to $v^* = -\frac{1}{r_0} B_0' P^\lambda$ and $\bar{x}_{v^*} = \bar{x}^\lambda$, where P^λ and $\bar{x}^\lambda$ are defined in (21)-(22) for a fixed point λ of F_u. It remains to show that F_u has a unique fixed point λ_*. Let λ and λ' be two distinct fixed points of F_u. Then, λ and λ' are respectively the fixed points of $s \mapsto F(s, y^\lambda)$ and $s \mapsto F(s, y^{\lambda'})$, where F is defined above (11) and y^λ in (21). Following Lemma 3, λ and λ' belong to $(0,1)$. But, $\beta' x^0 - \delta$ has a uniform distribution, which implies that F_u has a shape similar to that of the cumulative distribution function of a uniform distribution. Thus, all the real numbers in the interval $[0,1]$ are fixed points of F_u. In particular, $\lambda = 0$ is a fixed point of $s \mapsto F(s, y^{\lambda=0})$. This leads to a contradiction and shows that F_u has a unique fixed point.

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